Relational Structure of the Bekenstein–Hawking Entropy: Quantized Accretion and Evaporation, the Page Curve and the Entropy of Kerr Black Holes

Guoxing Gu

*Independent Researcher, Edmonton, Alberta, Canada.
ORCID: 0009-0006-0118-806X.

Corresponding author(s). E-mail(s): guoxing.gu@oseeds.ca;

Abstract

The Bekenstein–Hawking entropy $S = k_B A/(4\ell_p^2)$ of a Schwarzschild black hole specifies only the total amount of entropy and does not reveal what the entropy consists of. In a previous paper, taking the capsule E_c as the counting unit, we showed that the entropy of a Schwarzschild black hole of energy NE_P decomposes exactly as $S/k_B = 4\pi N^2 = 4\pi[N + N(N-1)]$: one self term per capsule and two ordered relations per pair of capsules, each term with weight 4π and no many-body terms; the coefficient $1/4$ comes from the minimal area cell $(2\ell_p)^2$ of the $2d_m$ axiom. Starting from this decomposition, we obtain results in three areas.
(i) Quantized accretion and evaporation. The Hawking temperature, heat capacity and Smarr relation are expressed in the integer N; the discrete temperatures for absorbing and emitting one capsule are $T_H \cdot 2N/(2N \pm 1)$, whose harmonic mean is the Hawking temperature. The entropy is superadditive, what is additive being $\sqrt{S}$, which gives counting interpretations of the area theorem, the bound on gravitational-wave energy, the impossibility of bifurcation and the no-hair theorem. Accretion and evaporation are quantized at two levels: energy changes in whole packs of E_P and area in sub-steps of $A_o = 16\pi\ell_p^2$, each sub-step having tunnelling probability $e^{-4\pi}$, independent of N; the horizon accepts only unpacked energy, and packing is completed at the inner interface $R_i = R_s/2$, with registration on the horizons. Temperature matching requires last-in-first-out, so that the history returns with the radiation in the reverse order of packing. The smallest black hole forms from two combining E_c, and evaporation ends with the unpacking $2 \to 1$; the temperature staircase is significant only for $N \lesssim 10^2$.
(ii) The Page curve. Conservation of relational entropy together with the unitarity of the overall evolution gives $S_{BH}/S_0 = (1-x)^2$ and $S_{out}/S_0 = x(2-x)$, with the information Page point $x_P = 1 - 1/\sqrt{2} \approx 0.293$; summing the entropy released pack by pack gives the same result independently, without island or

replica-wormhole constructions. The thermodynamic Page point, defined with the coarse-grained radiation entropy, moves earlier, to about 0.23–0.24, because of the entropy production of evaporation. The Page curve gives the amount of information; last-in-first-out gives its content.
(iii) The entropy of Kerr black holes. The entropies of the two horizons are obtained independently from their areas: the outer horizon carries the irreducible part $4\pi k_B N_{irr}^2$, the inner horizon the rotational part together with its cross relations with the irreducible part, and their sum is $4\pi k_B N^2$. Writing $N_{irr} = N\cos\omega$ gives $\sin 2\omega = \chi$, and the areal radii of the two horizons, $R_s\cos\omega$ and $R_s\sin\omega$, are the legs of a right triangle inscribed in the horizon circle, the geometric form of Christodoulou's irreducible-mass relation. The 29.3% upper bound on the extractable rotational energy corresponds to the balance point of the two horizon entropies, equal in value to the information Page point. The difference of the areal radii of the two horizons is not smaller than $2\ell_p$; $2\ell_p$ is the lower limit, and crossing it means breaking: the information registered on the two horizons mixes and can no longer be distinguished, leading to structural breakdown; constrained by the integer solutions, black holes with $N \leq 4$ do not rotate.

Keywords: Black-hole entropy, Bekenstein–Hawking entropy, Relational information, Discrete Hawking temperature, Superadditivity, Two-level quantization, Last in first out, Page curve, No-hair theorem, Kerr black hole, Irreducible mass

1 Introduction

Bekenstein proposed that a black hole has an entropy proportional to its horizon area [1], and Hawking showed that a black hole radiates with a thermal spectrum, thereby fixing the coefficient [2]:

$$S_{BH} = \frac{k_B A}{4\ell_p^2} \tag{1}$$

where A is the horizon area and $\ell_p = \sqrt{\hbar G/c^3}$ is the Planck length. Together with the four laws of black-hole mechanics of Bardeen, Carter and Hawking [3], Eq. (1) makes the black hole a thermodynamic system.

A large body of subsequent work has sought to reproduce the coefficient of Eq. (1) microscopically, for example the string-theory counting of microstates of extremal black holes [4], the loop-quantum-gravity counting of horizon punctures [5, 6], entanglement entropy [7, 8] and the "brick-wall" model [9]; among these, only the string-theory counting for extremal black holes contains no adjustable parameter, while the others fix the coefficient through the Immirzi parameter or a cutoff length. Equation (1), however, gives only the total amount of entropy and leaves four questions that it cannot answer by itself.

First, what does the entropy consist of? The area law means that the entropy scales as the square of the energy, whereas the entropy of an ordinary thermodynamic system is additive and scales with volume. When two black holes merge, the entropy

is strictly superadditive; Hawking's area theorem and the theorem that black holes cannot bifurcate [10–12] are geometric theorems of classical general relativity, derived from energy conditions and the causal structure of the horizon, and lack an interpretation in terms of counting; the no-hair theorem asserts that a merged black hole keeps no structure of its progenitors, and likewise lacks a counting interpretation; once Hawking radiation is included, the area theorem gives way to the generalized second law [1].

Second, does the entropy have a discrete structure? Bekenstein and Mukhanov proposed that the horizon area is quantized in equal steps [13, 14]. If the energy of a black hole changes in integer units, how a continuous Hawking temperature and a smooth thermal spectrum arise, and what the beginning and the end of evaporation look like, require a concrete counting to answer.

Third, how does information return? If the evolution is unitary (i.e. information is conserved), the entanglement entropy of the radiation must first rise and then fall, giving the Page curve [15, 16]. Equation (1) states only the total amount of information entropy, not what carries the information or in what order it returns with the radiation.

Fourth, how is the entropy of a rotating black hole distributed? A Kerr black hole has an inner and an outer horizon, and its entropy is usually counted from the area of the outer horizon alone, i.e. the irreducible part. Rotational energy can be extracted through the Penrose process; where it stands in the entropy, what the inner horizon carries, and where the upper limit of rotation lies are not answered by Eq. (1) itself.

This paper takes Eq. (1) as its basis; its coefficient $1/4$ can be read off from the minimal area cell of the $2d_m$ axiom described below (Section 2.1). Combined with the $2d_m$ axiom established in our previous paper [17], which constructs the dark-energy capsule unit E_c of inner diameter $D_i = 2\ell_p$ and energy E_P, it is shown that, with the number N of E_c capsules as the counting unit, the entropy of a Schwarzschild black hole of energy NE_P decomposes exactly into N self terms and $N(N-1)$ ordered pairwise relational terms, $S_{BH}/k_B = 4\pi N^2 = 4\pi N + 4\pi N(N-1)$, each term with weight 4π and no many-body remainder; taking this decomposition as the starting point, we answer the four questions above from different angles. Using only the counting in N, R_s and thermodynamic identities, we reveal content that the Bekenstein–Hawking formula cannot display. The results fall into three areas: the first two questions are answered by quantized accretion and evaporation (Sections 3–6), the third by the Page curve (Section 6.4) and the fourth by the entropy of Kerr black holes (Section 8).

Section 2 summarizes the counting basis, gives the weight of a relation and defines capsules as "intrinsically identical but relationally distinguishable". Section 3 expresses the known thermodynamic quantities in the integer N, gives the discrete absorption and emission temperatures, and discusses the mass at which a black hole is in equilibrium with the cosmic microwave background. Section 4 discusses the superadditivity of entropy and gives counting interpretations of the area theorem, the bound on gravitational-wave energy, the impossibility of bifurcation and the no-hair theorem. Section 5 uses additivity to distinguish relational entropy from thermal entropy and gives a unified reading of order and disorder of relational entropy. Section 6 discusses

accretion and evaporation: Section 6.1 gives the two-level quantization, the species of emitted particles and the boundary between packs and non-packs; Section 6.2 treats last-in-first-out and the packing order in a merger; Section 6.3 treats the formation and the end of the smallest black hole; and Section 6.4 obtains the Page curve from the conservation of relational entropy and from the pack-by-pack sum, with a comparison with the island approach. Section 7 compares the framework with other approaches in three respects: area steps, the definition of the end stage of evaporation and the temperature staircase, and logarithmic corrections. Section 8 extends the entropy counting to Kerr black holes: the geometry of the two horizons is first determined from the irreducible mass and the total energy; the distribution of entropy over the two horizons and its trigonometric geometry, the existence criterion for black holes and the Kerr temperature are then given; finally the stability of the Cauchy horizon is discussed. Section 9 concludes.

2 Counting basis

2.1 Black-hole entropy and the capsule number N

Denote the Planck length, Planck energy and Planck mass by

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}}, \qquad E_P = \sqrt{\frac{\hbar c^5}{G}}, \qquad m_P = \sqrt{\frac{\hbar c}{G}} \tag{2}$$

The ratio of the first two is

$$\frac{E_P}{\ell_p} = \sqrt{\frac{\hbar c^5}{G} \cdot \frac{c^3}{\hbar G}} = \sqrt{\frac{c^8}{G^2}} = \frac{c^4}{G}, \qquad \text{i.e.} \qquad E_P = \frac{c^4}{G}\,\ell_p \tag{3}$$

In our $2d_m$ axiom paper [17], the capsule E_c has inner diameter $D_i = 2\ell_p$ and outer diameter $D_o = 4\ell_p$, i.e. inner-interface radius $R_i = \ell_p$ and outer-interface radius $R_o = 2\ell_p$, and energy E_P. By Eq. (3), the Schwarzschild radius of E_P is $2GE_P/c^4 = 2\ell_p = R_o$: the outer interface lies exactly at the Schwarzschild radius of the Planck energy. The inner-interface radius $R_i = GE_P/c^4 = \ell_p$ is given by the force balance of the spacetime interfacial tension. Extending the interfacial force balance of Ref. [17] to black holes, the push $E/(4R)$ of the internal energy E at radius R balances the total pull $c^4/(4G)$ of the two interfaces, giving $R_i = GE/c^4$; for a black hole of mass M, $R_i = GM/c^2 = N\ell_p = R_s/2$, on which the push is exactly the maximum force $c^4/(4G)$ [17]. Packing and unpacking in a black hole take place at R_i, and the horizons do the registering. The inner interface R_i is the packing interface defined by interfacial force balance, not the inner (Cauchy) horizon of a Kerr black hole.

Let the energy of a Schwarzschild black hole be the sum of N capsule energies, $E = NE_P$. Its Schwarzschild radius is then

$$R_s = \frac{2GE}{c^4} = \frac{2GNE_P}{c^4} = N \cdot \frac{2GE_P}{c^4} = NR_o \tag{4}$$

and the horizon area is

$$A_H = 4\pi R_s^2 = 4\pi N^2 (2\ell_p)^2 = N^2 \cdot 16\pi\ell_p^2 = N^2 A_o, \qquad A_o \equiv 16\pi\ell_p^2 \tag{5}$$

A_o is the area of the outer interface of a single capsule E_c. Substituting into Eq. (1), the black-hole entropy S (in nats, in units of k_B) is

$$\frac{S}{k_B} = \frac{A_H}{4\ell_p^2} = \frac{N^2 \cdot 16\pi\ell_p^2}{4\ell_p^2} = 4\pi N^2 \tag{6}$$

The coefficient $1/4$ in Eq. (1) has a geometric origin in the $2d_m$ axiom [17]: a relation requires at least two minimal units, so the minimal relational span is $2d_m = 2\ell_p$ and the corresponding minimal area cell is $(2\ell_p)^2 = 4\ell_p^2$; no smaller area cell exists, and each cell carries one relation, i.e. 1 nat of information. The horizon area A contains $A/(4\ell_p^2)$ such cells, which gives Eq. (1); the 4 in the coefficient is exactly the square of the 2 in the minimal span $2\ell_p$. On this reading, the outer-interface area of a single capsule, $A_o = 16\pi\ell_p^2$, contains exactly 4π minimal area cells, consistent with the relation weight 4π given in Section 2.2. The area cells here are counting units and do not refer to a geometric shape. This origin agrees with the coefficient that Hawking fixed from the radiation temperature [2]; the present paper still takes Eq. (1) as its basis and does not repeat that argument.

Splitting N^2 into $N + N(N-1)$ gives the identity

$$\frac{S}{k_B} = 4\pi N^2 = 4\pi N + 4\pi N(N-1) = 4\pi\left[P(N,1) + P(N,2)\right] \tag{7}$$

where $P(N,k) = N!/(N-k)!$ is the number of k-permutations of N distinguishable capsules, $P(N,1) = N$ and $P(N,2) = N(N-1)$. The first term is the self terms of the N capsules; the second is the number of ordered pairwise relations among them. The identity has no remainder: the black-hole entropy consists entirely of self terms and ordered pairwise relations and contains no term in which three or more capsules take part simultaneously. $N = 1$ is E_c itself, with a self term and no relation; it is not a black hole, and the black-hole sequence starts at $N = 2$ [17].

Equation (7) gives a counting interpretation of Eq. (1): the black-hole entropy is exactly the sum of all capsule self terms and all ordered pairwise relations, with nothing missing and nothing extra. If three capsules could take part simultaneously, the entropy would acquire an extra term $P(N,3) = N(N-1)(N-2) \propto N^3$, contradicting the N^2 scaling of Eq. (1); relations are therefore established only between pairs of capsules. Note that Eq. (7) is a decomposition given on the premise that Eq. (1) holds; it provides a new interpretation of Eq. (1).

2.2 The weight of a relation: one A_o per relation

By Eq. (1), each area A_o on the horizon corresponds to the entropy

$$\frac{S_o}{k_B} = \frac{A_o}{4\ell_p^2} = \frac{16\pi\ell_p^2}{4\ell_p^2} = 4\pi \tag{8}$$

By Eq. (5), adding one capsule increases the horizon area by

$$A(N+1) - A(N) = \left[(N+1)^2 - N^2\right] A_o = (2N+1)A_o \tag{9}$$

On the other hand, adding one capsule adds exactly one self term plus $2N$ ordered relations between the new capsule and the existing N capsules (two directions per pair), $2N+1$ terms in all. The number of area cells in the increment matches the number of new terms one to one: each self term and each relation occupies one A_o on the horizon and corresponds to the entropy $4\pi k_B$. The weight of a relation is therefore the same as that of a self term, namely 4π.

Physically, the self term is itself the relation between the pair of minimal units inside a capsule (the inner diameter of a capsule is $2d_m$) [17], and belongs to the same kind of elementary relation as the relations between capsules; the difference is only whether it is "built in" or "newly established". The entropy thus consists of a single type of elementary contribution, namely relations, each worth 4π.

2.3 Intrinsic identity and relational distinguishability

Counting by permutations rather than combinations requires the capsules to be mutually distinguishable. If the E_c were entirely identical, (i, j) and (j, i) could not be told apart, ordered pairs would degenerate into unordered pairs, and one would be led to the Gibbs paradox: the count for identical particles must be divided by $N!$.

The resolution is to distinguish two kinds of "sameness".

Intrinsic identity. Every capsule E_c carries information: the minimal unit is $d_m = \ell_p$, while the inner diameter of a capsule is $2d_m$, so a capsule already contains a relation between a pair of minimal units, which is the self term 4π [17]. But this information is the same in every capsule. By the $2d_m$ axiom, the span $2d_m$ is the unique minimal realizable configuration [17], with no adjustable degree of freedom; the energy E_P, the geometric scales $2\ell_p$ and $4\ell_p$ and the self term 4π are therefore identical for all capsules. Carrying information and being mutually distinguishable are two different things: from intrinsic content alone, one capsule cannot be told apart from another.

Relational distinguishability. Inside a black hole, each capsule can be labelled by the number N of quantum capsules the black hole had accreted when it was packed (the N-th pack), and by the wavelength λ_N of the energy shares accumulated at the horizon at that time (Eq. (37)). These labels are state variables of the black hole's own history and do not depend on the time coordinate of an external observer. All capsules in a black hole are therefore pairwise distinguishable. This is the same as in a crystal: atoms of the same species are intrinsically identical but distinguishable by

lattice site, and the entropy of a solid is not divided by $N!$. The packing temperature and wavelength are properties of the sequence number (position), not of the capsule itself; capsules are intrinsically identical, and moving one to another position only requires re-registering; after a merger the temperature and wavelength of each pack are fixed anew by its new sequence number (Section 6.2).

Beyond distinguishability, direction is also needed. The relations listed in Section 2.1 of Ref. [17], such as left and right or front and back, are themselves directed: i relative to j and j relative to i are two different relations. In a black hole, "front and back" is realized as the "before and after" of packing. The number of identifiable relations is therefore the number of permutations $P(N, 2)$, not the number of combinations.

Free capsules in the cosmic vacuum background have not been packed and have no such relations among them; exchanging any two changes nothing, so they are indistinguishable and carry no inter-capsule relational entropy.

3 Thermodynamic quantities in terms of N and discrete temperatures

3.1 Known thermodynamic quantities in terms of N

The Hawking temperature, heat capacity and Smarr relation of a Schwarzschild black hole are known results [2, 18, 19]. Rewritten in terms of N using $E = NE_P$ and Eq. (6), for direct use below:

$$k_B T_H = \frac{E_P}{8\pi N} = \frac{\hbar c^3}{8\pi G M} \tag{10}$$

(using $E_P/N = E_P m_P/M = m_P^2 c^2/M$ and $m_P^2 = \hbar c/G$; Eq. (10) also follows directly from Eq. (6) and the first law of thermodynamics: $\mathrm{d}E = E_P\,\mathrm{d}N$ and $\mathrm{d}S = 8\pi k_B N\,\mathrm{d}N$, so $1/T_H = \mathrm{d}S/\mathrm{d}E = 8\pi k_B N/E_P$, i.e. $k_B T_H = E_P/(8\pi N)$; if Eq. (6) is regarded as a counting result in the $2d_m$ reading of Section 2.1, the Hawking temperature is its corollary),

$$C = \frac{\mathrm{d}E}{\mathrm{d}T} = -8\pi k_B N^2 = -2S \tag{11}$$

$$E = 2TS \tag{12}$$

The advantage of expressing them in N is that entropy and temperature are both functions of the integer N and can be computed capsule by capsule; this gives the discrete temperatures of the next subsection.

3.2 Discrete temperatures: absorption and emission temperatures

N is an integer, so one can compute capsule by capsule without taking the continuum limit.

When one capsule is absorbed ($N \to N+1$, denoted by the subscript $+$) or emitted ($N \to N-1$, denoted by the subscript $-$), $\Delta E = \pm E_P$, and

$$\Delta S_+ = S(N+1) - S(N) = 4\pi k_B \left[(N+1)^2 - N^2\right] = 4\pi k_B (2N+1) \tag{13}$$

$$\Delta S_- = S(N-1) - S(N) = 4\pi k_B \left[(N-1)^2 - N^2\right] = -4\pi k_B (2N-1) \tag{14}$$

Whether absorbed or emitted, the magnitude of the entropy change equals the one self term of the capsule entering or leaving, plus its ordered relations with the remaining capsules in the black hole (two directions per pair). On absorption the remaining capsules are the original N, giving $1+2N$ terms; on emission they are the $N-1$ that stay, giving $1+2(N-1) = 2N-1$ terms.

Defining the discrete temperature by $T = \Delta E/\Delta S$ (ΔE and ΔS are both positive on absorption and both negative on emission, so $T > 0$):

$$k_B T_N^+ = \frac{E_P}{4\pi(2N+1)}, \qquad k_B T_N^- = \frac{E_P}{4\pi(2N-1)} \tag{15}$$

where T_N^+ is the absorption temperature and T_N^- the emission temperature. Compared with Eq. (10):

$$\frac{T_N^\pm}{T_H} = \frac{E_P/\left[4\pi(2N \pm 1)\right]}{E_P/(8\pi N)} = \frac{8\pi N}{4\pi(2N \pm 1)} = \frac{2N}{2N \pm 1} \tag{16}$$

so that $T_N^+ < T_H < T_N^-$. The sum of their reciprocals is

$$\frac{1}{T_N^+} + \frac{1}{T_N^-} = \frac{4\pi k_B \left[(2N+1) + (2N-1)\right]}{E_P} = \frac{16\pi k_B N}{E_P} = \frac{2}{T_H} \tag{17}$$

i.e. the continuous Hawking temperature is exactly the harmonic mean of the absorption and emission temperatures. Their deviation from T_H is of order $1/(2N)$: negligible for large N, appreciable for small N. For the smallest black hole, $N = 2$ ($E_P/k_B = 1.417 \times 10^{32}$ K): $T_H = 2.82 \times 10^{30}$ K, $T_2^+ = \frac{4}{5} T_H = 2.25 \times 10^{30}$ K and $T_2^- = \frac{4}{3} T_H = 3.76 \times 10^{30}$ K.

The absorption and emission temperatures can also be written as values of the continuous Hawking temperature at half-integers. By Eq. (10), $k_B T_H(N \pm \frac{1}{2}) = E_P/[8\pi(N \pm \frac{1}{2})] = E_P/[4\pi(2N \pm 1)]$, so that $T_N^\pm = T_H(N \pm \frac{1}{2})$: the discrete absorption and emission temperatures are exactly the Hawking temperatures at the midpoints between adjacent levels. This is not an approximation; it follows from the entropy being exactly quadratic in N.

3.3 Temperature and the CMB-equilibrium mass

By Eq. (10), $T_H = \hbar c^3/(8\pi G k_B M) = 1.227 \times 10^{23}\ \mathrm{K\,kg}/M$; in Table 1, $N = M/m_P$ with $m_P = 2.176 \times 10^{-8}$ kg. Table 1 lists several representative values.

The mass in equilibrium with the cosmic microwave background follows from $T_{CMB} = 1.227 \times 10^{23}\ \mathrm{K\,kg}/M$: $M = 1.227 \times 10^{23}/2.7255 \approx 4.50 \times 10^{22}$ kg, about 0.6

Table 1 Hawking temperature

Object	Mass	N	T_H
Smallest black hole	4.35×10^{-8} kg	2	2.82×10^{30} K
In equilibrium with the CMB	4.50×10^{22} kg	2.07×10^{30}	2.725 K
One solar mass	1.989×10^{30} kg	9.14×10^{37}	6.17×10^{-8} K
10^9 solar masses	1.989×10^{39} kg	9.14×10^{46}	6.17×10^{-17} K

lunar masses, with a horizon radius of about 0.07 mm. By Eq. (11) the heat capacity is negative and this equilibrium is unstable [18]: absorbing a little more makes the black hole larger and colder, so it absorbs still more; emitting a little more makes it smaller and hotter, accelerating evaporation. All astrophysical black holes are far colder than the microwave background and at present only absorb on balance. Whether a black hole absorbs or evaporates on balance is decided by comparing T_H with the microwave background temperature T_{bg}, which is thermodynamics; how fast the process runs is decided by dynamics: the evaporation time is proportional to M^3 [20], and primordial black holes heavier than about 5×10^{11} kg have lifetimes exceeding the age of the universe [21]. The microwave background temperature falls with cosmic expansion as $T_{bg} \propto 1/a$, so the equilibrium mass $M_{eq} \propto 1/T_{bg}$ grows; an astrophysical black hole begins to evaporate on balance only when M_{eq} has grown beyond its own mass, i.e. when the microwave background has cooled below T_H.

4 Superadditivity of black-hole entropy: merger and recursion

4.1 General recursion: what is additive is $\sqrt{S}$

Let a large black hole be formed by the merger of several black holes, progenitor i having contained N_i capsules with entropy $S_i = 4\pi k_B N_i^2$. Since the entropy depends only on the total capsule number,

$$S = 4\pi k_B \Big(\sum_i N_i \Big)^2 = \sum_i 4\pi k_B N_i^2 + \sum_{i \neq j} 4\pi k_B N_i N_j \tag{18}$$

Note that $4\pi k_B N_i N_j = \sqrt{4\pi k_B N_i^2 \cdot 4\pi k_B N_j^2} = \sqrt{S_i S_j}$, so

$$S = \sum_i S_i + \sum_{i \neq j} \sqrt{S_i S_j} = \Big(\sum_i \sqrt{S_i} \Big)^2 \tag{19}$$

The first term is the entropy the progenitors had, the second the relations between capsules from different progenitors (hereafter called cross relations): between progenitors i and j there are $2N_iN_j$ of them, corresponding to the two terms (i, j) and (j, i) of the sum, each of value $4\pi k_B N_i N_j = \sqrt{S_i S_j}$, the geometric mean of the two entropies. This is not a relation between two progenitors but the sum of the relations among N_iN_j pairs of capsules. What is additive is $\sqrt{S}$, not S.

If M equal black holes merge, each containing n capsules, then $N = Mn$, and writing $S_n = 4\pi k_B n^2$,

$$S = 4\pi k_B M^2 n^2 = S_n M^2 = S_n \left[M + M(M-1)\right] = S_n \left[P(M,1) + P(M,2)\right] \quad (20)$$

The structure of Eq. (7) reappears unchanged one level up, with the unit changed from $4\pi k_B$ (one capsule) to S_n (the entropy of one progenitor). Capsules, black holes and black holes formed by merging black holes have the same counting form, each level containing only self terms and pairwise relations; but this is an isomorphism of counting only, and the progenitors do not survive as units after the merger.

A black hole cannot contain complete sub-black-holes. If a progenitor survived inside the merged black hole as a complete sub-black-hole, it would be an identifiable relational unit: besides the capsule terms of Eq. (18) there would be at least its own self term as a unit and its ordered relations with the other units, and the total entropy would exceed $4\pi k_B(\sum_i N_i)^2$, i.e. the entropy fixed uniquely by the area (Eq. (1)); but the area leaves no room. For example, $N_A = 2$, $N_B = 3$ and $N_C = 5$ merge into $N = 10$, with entropy $10^2 = 100$ (in units of $4\pi k_B$), exactly the sum of 10 self terms and 90 ordered relations; if A and B survived as sub-black-holes, there would be at least their two self terms and the 2 ordered relations between them, and the total entropy would become $100 + 1 + 1 + 2 = 104 > 100$. A black hole therefore contains only capsules and the pairwise relations among them; the progenitors do not survive as independent units, and the area, temperature and entropy depend only on the total capsule number, which is a counting statement of the no-hair theorem. The dissolution is completed during the merger: the interfaces of the smaller black hole disappear inside the common outer interface; how the capsules are re-registered in a merger is described in Section 6.2.

Merging without splitting. Splitting a black hole into $N_1 + N_2$ is exactly the reverse of a merger. By Eq. (19), when two parts combine, $S - S_1 - S_2 = 2\sqrt{S_1 S_2} = 8\pi k_B N_1 N_2$; on splitting, all $2N_1N_2$ relations between the capsules of the two parts would be removed and the entropy would decrease by $8\pi k_B N_1 N_2$, which the second law of thermodynamics forbids. This corresponds to the theorem of classical general relativity that black holes cannot bifurcate [11, 12]. N can decrease only step by step through radiation (Section 6), never by splitting in two directly.

4.2 Entropy increase in a merger

When N_1 capsules and N_2 capsules, which previously had no inter-capsule relations with each other, combine into one system, neglecting the energy carried off by gravitational waves,

$$\begin{aligned} \Delta S &= S(N_1 + N_2) - S(N_1) - S(N_2) \\ &= 4\pi k_B \left[(N_1 + N_2)^2 - N_1^2 - N_2^2\right] = 8\pi k_B N_1 N_2 > 0 \end{aligned} \quad (21)$$

The extra horizon entropy is exactly the cross relations between the capsules of the two parts: N_1N_2 pairs, two directions per pair, each worth 4π. Two E_c combining into the smallest black hole ($N_1 = N_2 = 1$) and the merger of two black holes ($N_1, N_2 \geq 2$) have

the same counting form but different origins: free E_c have no relations with one another (Section 2.3), and the 2 cross relations are newly established on combination; two black holes first establish group relations during their approach, which are converted into cross relations in the merger (see below). Neglecting the energy carried off by gravitational waves, $N_f = N_1 + N_2$ and

$$N_f^2 = (N_1 + N_2)^2 = N_1^2 + N_2^2 + 2N_1N_2 > N_1^2 + N_2^2 \tag{22}$$

The horizon entropy increases strictly, and the increase $8\pi k_B N_1 N_2$ is exactly the $2N_1N_2$ cross relations.

In a real merger, part of the energy leaves as gravitational waves and part becomes rotational energy of the new black hole. By Section 8.2, a rotating black hole is counted by its irreducible mass; writing $N_{irr,f}$ for the irreducible capsule number of the new black hole, the number of capsules leaving the irreducible part is

$$N_u = N_1 + N_2 - N_{irr,f} \tag{23}$$

Before the merger, the number of relations of the two black holes is $N_1^2 + N_2^2$; during their approach the two also establish group relations: taking each black hole as one unit, the number of the ordered relations between the units is $P(2,2) = 2$, each weighted by the geometric mean of the two entropies, $\sqrt{S_1 S_2} = 4\pi k_B N_1 N_2$ (the same as the weight of the cross terms in Eq. (19)), $8\pi k_B N_1 N_2$ in all, equal in amount to the cross relations. The group relations exist in the bound binary system, outside the horizons, and are not counted in the horizon entropy; when the common outer interface forms, they are converted into $2N_1N_2$ cross relations and counted in the horizon entropy of the new black hole. The total number of relations before the merger is therefore $N_1^2 + N_2^2 + 2N_1N_2 = (N_1 + N_2)^2$ (in units of $4\pi k_B$). After the merger, $N_{irr,f}^2$ relations remain in the new black hole, and the other X relations leave the irreducible part with the N_u departing capsules, borne by the gravitational waves and the rotational energy. By conservation of information, $(N_1 + N_2)^2 = N_{irr,f}^2 + X$, so

$$\begin{aligned} X &= (N_1 + N_2)^2 - N_{irr,f}^2 = (N_1 + N_2 - N_{irr,f})(N_1 + N_2 + N_{irr,f}) \\ &= N_u(N_1 + N_2 + N_{irr,f}) \end{aligned} \tag{24}$$

By Hawking's area theorem [10, 11], $N_{irr,f}^2 \geq N_1^2 + N_2^2$, i.e. $N_{irr,f} \geq \sqrt{N_1^2 + N_2^2}$. Substituting into Eq. (23):

$$N_u \leq N_1 + N_2 - \sqrt{N_1^2 + N_2^2} \tag{25}$$

Equivalently, $X \leq 2N_1N_2$: the number of relations removed from the irreducible part does not exceed the number of group relations before the merger. For equal masses $N_u \leq (2 - \sqrt{2})N_1$. The number of capsules carried off by gravitational waves, $N_{GW} = E_{GW}/E_P$, is part of N_u (the other part being rotational energy), so $N_{GW} \leq N_u$, and Eq. (25) also gives the upper bound on the gravitational-wave energy, which is Hawking's bound [10]. The area theorem thus acquires a counting interpretation: the

number of relations $N^2_{irr,f}$ in the new black hole after the merger is no less than the sum $N_1^2 + N_2^2$ of the relations the two black holes had, the difference being $N^2_{irr,f} - N_1^2 - N_2^2 = 2N_1N_2 - X \geq 0$. The X relations removed from the irreducible part are those carried by the N_u departing capsules in the merged sequence (their self terms and their relations with the other capsules), and these capsules are unpacked from the tail of the sequence of the new black hole (Section 6.2). X consists of two parts, $X = [(N_1 + N_2)^2 - N_f^2] + [N_f^2 - N^2_{irr,f}]$: the first leaves the black hole with the gravitational-wave energy, and the second remains in the new black hole as rotational relations registered on the inner horizon (Section 8.2).

5 Relational entropy, statistical entropy and the observer

5.1 Relational entropy and thermal entropy: definitions and a quantitative comparison

Relational entropy (also called ordered entropy or information entropy in this paper) measures the ordered arrangements already established, i.e. the number of identifiable directed relations, each relation in Eq. (7) counting $4\pi k_B$. **Thermal entropy** (also called disordered entropy or matter entropy here) measures disordered arrangements, i.e. the number of indistinguishable microscopic configurations in one macrostate, $S_{th} = k_B \ln W$. The two are different in nature but measured in the same unit; they convert into each other in the packing of accretion and the unpacking of radiation (Section 6.2), and their sum obeys the second law of thermodynamics. This paper does not redefine thermodynamics; it only decomposes the entropy and provides an accounting of their conversion, in line with the generalized second law [1].

Within one system the two can be distinguished by additivity. When two independent subsystems 1 and 2 are joined, thermal entropy is additive and relational entropy is superadditive:

$$S_{th}(1+2) = S_{th}(1) + S_{th}(2) \tag{26}$$

$$S(1+2) = S(1) + S(2) + 8\pi k_B N_1 N_2 \tag{27}$$

Thermal entropy has no cross term; the extra cross term $8\pi k_B N_1 N_2$ of relational entropy is Eq. (21), the cross relations between the capsules of the two parts. The excess of the whole over the sum of the parts is therefore the relational part, which is operationally testable. An auxiliary criterion: relational entropy and temperature are both fixed uniquely by the structural variable N, so temperature is not an independent variable and cannot be changed at fixed structure; the relation $S \propto T^{-2}$ merely reflects their joint dependence on N. In thermal entropy, temperature is an independent variable, and the entropy changes with temperature at fixed structure.

It should be noted that black-hole entropy does enter the first law of black-hole thermodynamics, $\mathrm{d}E = T\,\mathrm{d}S$ [2, 3], and the Hawking temperature is tied to it through this relation, so relational entropy cannot be said to be unrelated to thermodynamics. The point of the distinction is that it is entropy existing in the form of relations, released as thermal entropy only on unpacking (Section 6.2).

The photon gas illustrates the difference quantitatively: if the interior of the horizon is regarded as an ordinary thermodynamic system of N energy shares and its states are counted in the volume, the result is entirely different. Let the energy be $E = NE_P$ and the volume $V = N^3 V_c$, where $V_c = \frac{4}{3}\pi(2\ell_p)^3 = \frac{32\pi}{3}\ell_p^3$ is the volume of a single capsule (the Euclidean equivalent volume defined by the areal radius). The energy density of black-body radiation is

$$\frac{E}{V} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} \tag{28}$$

Solving for the temperature:

$$(k_B T)^4 = \frac{15}{\pi^2}(\hbar c)^3 \frac{E}{V} = \frac{15}{\pi^2}(\hbar c)^3 \frac{NE_P}{N^3 \cdot \frac{32\pi}{3}\ell_p^3} = \frac{45}{32\pi^3}\left(\frac{\hbar c}{\ell_p}\right)^3 \frac{E_P}{N^2} \tag{29}$$

With $\hbar c/\ell_p = E_P$,

$$k_B T_{\text{gas}} = \left(\frac{45}{32\pi^3}\right)^{1/4} \frac{E_P}{\sqrt{N}} \approx 0.461\,\frac{E_P}{\sqrt{N}} \tag{30}$$

The entropy of black-body radiation is $S = 4E/(3T)$; substituting $E = NE_P$ and Eq. (30):

$$S_{\text{gas}} = \frac{4}{3}\cdot\frac{NE_P}{T_{\text{gas}}} = \frac{4}{3}\left(\frac{32\pi^3}{45}\right)^{1/4} N^{3/2} k_B \approx 2.889\, k_B N^{3/2} \tag{31}$$

Compared with Eq. (6), $S_{\text{gas}}/S = 2.889/(4\pi\sqrt{N}) \approx 0.230/\sqrt{N}$. For a solar-mass black hole ($N \approx 9.14 \times 10^{37}$), $S_{\text{gas}} \approx 2.5 \times 10^{57} k_B$, whereas the black-hole entropy is $S \approx 1.05 \times 10^{77} k_B$.

$$\frac{T_{\text{gas}}}{T_H} = \frac{0.461\, E_P/\sqrt{N}}{E_P/(8\pi N)} \approx 11.6\sqrt{N} \tag{32}$$

Again for a solar-mass black hole, $T_{\text{gas}} \approx 6.8 \times 10^{12}$ K, while $T_H \approx 6.17 \times 10^{-8}$ K. Note that T_{gas} and T_H are two different quantities: the former is the temperature of a hypothetical photon gas filling the volume inside the horizon, the latter the temperature at which the horizon radiates outward. Eq. (32) shows that the former far exceeds the latter: the hypothetical uniform photon-gas model fails to reproduce either the Hawking temperature or the Bekenstein–Hawking entropy.

High temperature and low entropy are not contradictory. At fixed energy, the entropy of thermal radiation, $S = 4E/(3T)$, is inversely proportional to temperature: the higher the temperature, the fewer and more energetic the photons carrying the same energy, and the lower the entropy. The black hole is the opposite case: its temperature is extremely low and, by Eq. (12), $S = E/(2T)$ is enormous.

The entropy of the photon gas scales as $N^{3/2}$, not N^2. This shows that the area law cannot come from counting particle states in the volume [9, 22]. Moreover, conventional thermal entropy depends on internal grouping (dissolving a subsystem increases the entropy of mixing), whereas by Section 4.1 relational entropy depends only on the

total capsule number, not on the origin of the capsules. Conventional counting treats E_c as identical particles and must divide by $N!$; relational counting distinguishes them by labels (Section 2.3).

5.2 Two readings of the same set of relations

The mainstream view holds that black-hole entropy is maximal and the black hole therefore the most disordered [23]; the reading of this paper is that all relations inside a black hole are established and it is therefore the most ordered. The two look opposite but are two readings of the same number.

Let X be the relational configuration inside the horizon and Y the information available to an observer. In information theory, the information entropy $H(X)$ of X, the conditional entropy $H(X|Y)$ of X given Y and their mutual information $I(X;Y)$ satisfy the identity [24] (all in nats and multiplied by k_B, so that they have the units of thermodynamic entropy)

$$H(X) = H(X|Y) + I(X;Y) \tag{33}$$

$H(X)$ is the relational capacity, which for a black hole is $S_{BH} = k_B A/(4\ell_p^2) = 4\pi k_B N^2$ from Eq. (1); $H(X|Y)$ is the part missing to the observer, and $I(X;Y)$ the part the observer holds.

External reading. By the no-hair theorem, an external observer (before collecting the historical information in the radiation) can measure only the mass M, angular momentum J and charge Q of a black hole and knows nothing about its internal relations: $I(X;Y) = 0$ and $H(X|Y) = S_{BH}$, which appears as complete disorder, i.e. Hawking thermal radiation.

Internal structural reading. The relations are completely fixed by the labels (Section 2.3): $H(X|Y) = 0$ and $I(X;Y) = S_{BH}$, which appears as complete order.

Define the disorder D as the fraction of the relational capacity that is missing to the observer:

$$D = \frac{H(X|Y)}{S_{BH}}, \qquad 1 - D = \frac{I(X;Y)}{S_{BH}} \tag{34}$$

$1 - D$ is the order. The external reading has $D = 1$, the internal reading $D = 0$. This has the same form as Landsberg's definitions of disorder as $S/S_{\max}$ and order as $1 - S/S_{\max}$ [25], and agrees with the view that "entropy is missing information" [26]. "Black-hole entropy is maximal, hence most disordered" is the external reading; "all relations are identified" is the internal reading; both correspond to the same S_{BH}, and there is no contradiction. Note that the "internal reading" is a statement about structure, not the result of a measurement by any actual observer.

6 Accretion and evaporation: quantization, last-in-first-out and the Page curve

6.1 Two-level quantization

Energy level. N is an integer and the energy changes in whole steps of E_P: each packing increases N by 1 and each unpacking decreases it by 1. This is the only

intrinsic fluctuation of a black hole. At each step the horizon radius changes by $\Delta R_s = \pm 2GE_P/c^4 = \pm 2\ell_p$.

Area level. By Eq. (9), when the black hole grows from N packs to $N+1$ packs the horizon area increases by $(2N+1)A_o$; conversely, from N packs to $N-1$ packs it decreases by $(2N-1)A_o$. The area increment $(2N+1)A_o$ of one packing consists of $2N+1$ sub-steps, each of one A_o, corresponding to one new self term or relation. Each sub-step $A_o = 16\pi\ell_p^2 = 4\pi \cdot (4\ell_p^2)$ is 4π of the area cells $4\ell_p^2$ commonly used in the literature (each corresponding to 1 nat) [1, 2], so each sub-step corresponds to the entropy $4\pi k_B$ (Eq. (8)).

One E_P is shared among $2N+1$ sub-steps, and the energy share of each sub-step is

$$\varepsilon_N = \frac{E_P}{2N+1} \tag{35}$$

Sub-steps are registration and energy-allocation units within a pack, not independent equilibrium Schwarzschild states; the energy of a pack is shared equally among its sub-steps. The temperature of a sub-step is still computed from the definition $T = \Delta E/\Delta S$ of Section 3.2. A sub-step has energy $\Delta E = \varepsilon_N$ and entropy $\Delta S = 4\pi k_B$ (one A_o), so

$$k_B T_N^+ = \frac{k_B\,\Delta E}{\Delta S} = \frac{\varepsilon_N}{4\pi} = \frac{E_P}{4\pi(2N+1)} \tag{36}$$

in agreement with $k_B T_N^+$ of Eq. (15): the temperature of each sub-step equals the absorption temperature of the whole pack, because both the energy and the entropy of the whole pack are $2N+1$ times those of a sub-step, leaving the ratio unchanged.

If the energy share ε_N is regarded as a single quantum, the relation $E = 2\pi\hbar c/\lambda$ between photon energy and wavelength gives its characteristic wavelength

$$\lambda_N = \frac{2\pi\hbar c}{\varepsilon_N} = 2\pi(2N+1)\frac{\hbar c}{E_P} = 2\pi(2N+1)\,\ell_p = 2\pi(R_s + \ell_p) \tag{37}$$

using $\hbar c/E_P = \ell_p$ and $R_s = 2N\ell_p$. λ_N is exactly the circumference at the midpoint as the horizon grows from $R_o(N)$ to $R_o(N+1)$; the same formula at the starting point $N = 0$ (from 0 to 1, i.e. the formation of a single E_c) gives $\lambda_0 = 2\pi\ell_p$, the inner circumference of E_c [17]. The characteristic wavelength of an energy share and the inner circumference of a capsule are given by the same formula. This is equivalent to Bohr's quantization condition $n\lambda = 2\pi r$ with $n = 1$: each energy share is exactly one wave "around the horizon". Bekenstein obtained area quantization along these lines from the adiabatic invariance of the area [13]. The larger N, the smaller the relative difference between adjacent steps and the more classical the behaviour of the black hole, in accordance with the correspondence principle. In reality this energy is emitted in the unpacking $N+1 \to N$ as thermal radiation at the emission temperature T_{N+1}^- (with $T_{N+1}^- \approx T_H$ for large N), not as a single photon. By Eq. (15), $k_B T_{N+1}^- = E_P/\{4\pi[2(N+1)-1]\} = E_P/[4\pi(2N+1)] = k_B T_N^+$: the emission temperature of this share equals its absorption temperature. The mean photon energy of black-body

radiation is

$$\langle E_\gamma \rangle = \frac{\pi^4}{30\,\zeta(3)}\, k_B T^-_{N+1} \approx 2.70\, k_B T^-_{N+1} \tag{38}$$

where $\zeta(3) \approx 1.202$. From Eq. (36) and the above, $\varepsilon_N = 4\pi k_B T^+_N = 4\pi k_B T^-_{N+1}$, so the number of photons emitted per sub-step is about

$$\frac{\varepsilon_N}{\langle E_\gamma \rangle} = \frac{4\pi k_B T^-_{N+1}}{2.70\, k_B T^-_{N+1}} \approx 4.65 \tag{39}$$

The corresponding photon wavelength is $2\pi\hbar c/\langle E_\gamma \rangle = (\varepsilon_N/\langle E_\gamma \rangle)\,\lambda_N \approx 4.65\,\lambda_N \approx 29\,R_s$. The energy share of one sub-step thus corresponds to the horizon scale $2\pi R_s$ and is emitted as about 4.65 thermal photons, each with a wavelength about 29 times the horizon radius; this is the typical wavelength of Hawking radiation.

Species of emitted particles. By the definition in the $2d_m$ axiom paper [17], unpacking emits massless particles, of which the photon is only the representative: nearly massless neutrinos are emitted as well, and so are gravitons if they exist. When the temperature rises to $k_B T \gtrsim mc^2$, the corresponding massive particles are also emitted [20, 21], matching the acceptance of ordinary matter by the horizon on packing. Taking the electron as the lightest charged massive particle, this begins at $N \lesssim 9.5 \times 10^{20}$ (Section 7); for the last few packs $k_B T \sim 10^{26}$ eV, far above the rest energies of all known particles, so the emission of all kinds of massive particles is highly likely. The particle species does not affect the energy ε_N or the entropy $4\pi k_B$ of a sub-step, only the number of particles: the mean energy of a massless fermion is about $3.15\, k_B T$, giving about $4\pi/3.15 \approx 3.99$ per sub-step. In this paper the particle numbers are counted with the photon as representative; Eqs. (39) and (52) both refer to photons.

Tunnelling probability. Parikh and Wilczek treated Hawking radiation as tunnelling across the horizon and, including energy conservation, obtained the emission probability [27]

$$\Gamma \approx e^{\Delta S/k_B} \tag{40}$$

where $\Delta S < 0$ is the decrease in black-hole entropy due to the emitted energy. For one sub-step the horizon area decreases by one A_o and, by Eq. (8), $\Delta S = -4\pi k_B$:

$$\Gamma_1 = e^{-4\pi} \approx 3.5 \times 10^{-6} \tag{41}$$

independent of N. If a whole pack of energy E_P were emitted at once as one quantum, the black hole would drop from N to $N-1$, and by Eq. (6)

$$\frac{\Delta S}{k_B} = 4\pi\left[(N-1)^2 - N^2\right] = 4\pi(1-2N) = -4\pi(2N-1) \tag{42}$$

so that

$$\Gamma_{\text{pack}} = e^{-4\pi(2N-1)} = \left(e^{-4\pi}\right)^{2N-1} = \Gamma_1^{2N-1} \tag{43}$$

The probability of emitting a whole pack at once equals the product of the probabilities of $2N-1$ sub-steps and is extremely suppressed for large N. In terms of tunnelling

probability, a single sub-step has an exponential advantage of $e^{8\pi(N-1)}$ over emitting a whole pack at once; even for the smallest black hole, $N = 2$, this ratio is about 8.2×10^{10}, so for comparable attempt frequencies sub-step emission dominates overwhelmingly. A pack of energy is therefore emitted not as one quantum but in $2N - 1$ sub-steps.

The boundary between packs and non-packs. Whether an energy share is self-bound can be judged by Thorne's hoop conjecture [28] (Schoen and Yau proved a rigorous form of it [29]): if the circumference of the region occupied by energy E is no greater than $2\pi R_s(E)$, where $R_s(E) = 2GE/c^4$ is its own Schwarzschild radius, the energy is bound by its own gravity; if the circumference is much greater than $2\pi R_s(E)$, the energy propagates as ordinary radiation. For a single quantum, taking the wavelength $\lambda = 2\pi\hbar c/E$ as this circumference scale, the ratio is

$$\frac{\lambda}{2\pi R_s(E)} = \frac{2\pi\hbar c/E}{2\pi \cdot 2GE/c^4} = \frac{1}{2}\frac{\hbar c^5}{G}\frac{1}{E^2} = \frac{1}{2}\left(\frac{E_P}{E}\right)^2 \tag{44}$$

For an E_c pack, $E = E_P$ and the ratio is $1/2 < 1$: the pack is self-bound. For a sub-step share, $E = \varepsilon_N = E_P/(2N+1)$ and the ratio is $(2N+1)^2/2 \gg 1$: the share is not self-bound and can propagate as radiation. Setting the ratio in Eq. (44) equal to 1 gives the critical energy for self-binding, $E_{\text{crit}} = E_P/\sqrt{2}$: a whole pack E_P lies above it, and every sub-step share (at most $E_P/3$) lies below it. Packing $N \to N+1$ thus assembles $2N+1$ shares that are not self-bound into one self-bound pack, and unpacking $N+1 \to N$ reverses this; packing compresses these shares within $2\ell_p$, which requires very strong gravity, available only at the inner interface R_i.

This gives two rules. (i) Packing: the outer interface (the horizon) accepts only unpacked energy (radiation and ordinary matter), accumulating it share by share in shares matched to the current temperature, and a full E_P is assembled into a pack at the inner interface R_i and registered on the horizons. E_c in the cosmic background, already packed, are not drawn in: the accretion rate of dark energy with $w = -1$ onto a black hole is zero [30], and a growing black hole merely pushes the E_c of the cosmic background aside. Rule (i) concerns accretion. A merger is different: both black holes are gravitating sources with $N \geq 2$ and fall into each other as matter; a common outer interface forms outside the two horizons, and the capsules of the smaller black hole enter with it as a whole, without passing through the share stage (Section 6.2). A single E_c is a dark-energy unit, accreted at zero rate, and does not enter in this way. (ii) Unpacking: a pack is decomposed into several shares at R_i, which pass through the channel between R_i and the outer horizon by tunnelling and are emitted at the outer horizon; this is Hawking radiation.

Two-level quantization answers the question "how can an integer N accommodate continuous Hawking radiation": at the area level the horizon changes almost continuously in sub-steps of A_o, while at the energy level it remains an integer on the scale of packs.

6.2 Last-in-first-out and mergers

Last in, first out. The last-in-first-out (LIFO) principle applies to the registration sequence and is constrained by temperature matching. By Eq. (10), $k_B T_H = E_P/(8\pi N)$: the temperature is inversely proportional to the number of packs, and the smaller the black hole the hotter it is. Packing proceeds in pack order, the 1st pack, the 2nd pack, ... entering in turn; the black hole grows and the packing temperature gradually falls; correspondingly, the share $\varepsilon_N = E_P/(2N+1)$ accumulated at the horizon during packing (Eq. (35)) decreases and the wavelength $\lambda_N = 2\pi(2N+1)\ell_p$ (Eq. (37)) lengthens. Packs that enter early therefore have high packing temperatures and short wavelengths, and packs that enter late have low packing temperatures and long wavelengths. By Eq. (15), the packing temperature of the $(N+1)$-th pack (packing $N \to N+1$) is

$$k_B T_N^+ = \frac{E_P}{4\pi(2N+1)} \tag{45}$$

with entropy change $\Delta S_+ = 4\pi k_B(2N+1)$ (Eq. (13)): the entering capsule brings one self term plus $2N$ ordered relations with the existing N capsules.

Unpacking is likewise constrained by temperature: a black hole can unpack only at its current temperature. When the evaporating black hole has returned to the state with $N+1$ packs, the next unpacking is $N+1 \to N$ (unpacking $N \to N-1$ with N replaced by $N+1$), and by Eq. (15) the emission temperature is

$$k_B T_{N+1}^- = \frac{E_P}{4\pi\,[2(N+1)-1]} = \frac{E_P}{4\pi(2N+1)} = k_B T_N^+ \tag{46}$$

exactly the packing temperature of the $(N+1)$-th pack. The entropy change is $\Delta S_- = -4\pi k_B[2(N+1)-1] = -4\pi k_B(2N+1)$: the departing capsule takes away one self term plus $2N$ ordered relations with the N capsules that remain, the same as on packing; the number of sub-steps $2N+1$, the share ε_N and the wavelength λ_N are also the same as on packing. In every state, therefore, the only pack whose temperature matches and which can be unpacked is the last one to have entered. As the black hole shrinks and its temperature rises again, the late, low-temperature packs leave first and the early, high-temperature packs leave last, the emitted wavelength shortening from long to short, exactly the reverse of the packing order: unpacking must be last in, first out, and the history returns with the radiation in reverse order. This gives a concrete picture of the return of information; the quantitative Page curve is given at the end of this section.

Merger: enclosure, dissolution and re-registration. A merger is not accretion. In accretion, energy crosses the horizon share by share and is packed at the inner interface R_i; in a merger, a common outer interface appears outside the horizons of the two black holes and encloses both as wholes. In numerical relativity the common apparent horizon indeed appears suddenly outside the two horizons, and the original horizons persist inside it for a while before disappearing (< 1 ms) [31]. Once the common outer interface has formed, everything inside it obeys the rules of the new black hole.

Let $N_1 \geq N_2$. The outer and inner horizons of the smaller black hole then disappear, i.e. it dissolves. Its interior consists entirely of E_c packs, and E_c is the smallest self-bound unit (by Eq. (44), the ratio is 1/2 for $E = E_P$), so the dissolution stops at the interfaces and at its identity as a unit, and does not reach the E_c themselves: the N_2 packs of the smaller black hole, together with the $N_2(N_2-1)$ ordered relations already among them, are incorporated as a whole. A merger therefore requires no unpacking or packing, only re-registration: the packs of the larger black hole keep their sequence numbers, and those of the smaller black hole are appended after them in their original order,

$$\begin{aligned} k &= j \qquad && (j\text{-th pack of the larger black hole}) \\ k &= N_1 + j && (j\text{-th pack of the smaller black hole}) \end{aligned} \tag{47}$$

By Eq. (15), the packing temperature of the k-th pack is $k_B T = E_P/[4\pi(2k-1)]$. The packing temperatures of the j-th pack of the smaller black hole before and after re-registration are

$$k_B T_j = \frac{E_P}{4\pi(2j-1)}, \qquad k_B T_{N_1+j} = \frac{E_P}{4\pi(2N_1+2j-1)} \tag{48}$$

Both decrease monotonically with j, so appending in the original order preserves the temperature ranking of the packs: the original 1st pack, the hottest, takes the first and hottest position of the appended segment, and the original N_2-th pack, the coldest, takes the last; each temperature is lowered by the factor $(2j-1)/(2N_1+2j-1) < 1$, consistent with the merger making the black hole colder. The packs are incorporated in order from hot to cold.

Unpacking from the tail. By Eqs. (23)–(25), after the merger $N_u = N_1 + N_2 - N_{irr,f}$ capsules leave the irreducible part. By last-in-first-out they leave the irreducible part in turn from the tail of the sequence (N_{GW} of them unpack and leave with the gravitational waves, and the rest are re-registered as the rotational part, Section 8.2), i.e. the packs of the smaller black hole that entered last. By Eq. (14), the pack at position k takes away on unpacking its self term and its ordered relations with the other $k-1$ packs, $2k-1$ in all. For the last N_u packs, i.e. packs $(N_{irr,f}+1)$ to (N_1+N_2), the relations leaving the irreducible part total, using $\sum_{k=1}^{n}(2k-1) = n^2$, $\sum_{k=N_{irr,f}+1}^{N_1+N_2}(2k-1) = (N_1+N_2)^2 - N_{irr,f}^2 = X$, in agreement with Eq. (24). These packs all come from the smaller black hole: by the area theorem $N_{irr,f}^2 \geq N_1^2 + N_2^2 > N_1^2$, so $N_{irr,f} > N_1$ and

$$N_u = N_1 + N_2 - N_{irr,f} < N_2 \tag{49}$$

The packs of the larger black hole are unaffected. Last-in-first-out thus holds at two levels: the packs unpacked are both the last registered in the new black hole and the last packed by the smaller black hole itself.

Examples. Table 2 gives the accounts of two mergers, GW150914 [32] and GW190814 [33]. The observed black-hole masses M are converted to capsule numbers by $N = M/m_P$; one solar mass contains $N_\odot \equiv M_\odot/m_P = 1.989 \times 10^{30}/(2.176 \times 10^{-8}) = 9.14 \times 10^{37}$ capsules. In the table, capsule numbers are in units of $N_\odot$, and relation numbers (in units of $4\pi k_B$) in units of $N_\odot^2$. N_1, N_2 and N_f are the capsule numbers of the primary, the secondary and the new black hole, N_f including rotational energy; χ_f is the dimensionless spin of the new black hole. The irreducible capsule number $N_{irr,f}$ of the new black hole follows from Christodoulou's irreducible-mass relation [34], which in capsule numbers reads $N_f^2 = N_{irr,f}^2 + \mathcal{J}^2/(4N_{irr,f}^2)$, with $\mathcal{J} = J/\hbar$ (Section 8.3). Multiplying both sides by $N_{irr,f}^2$ gives $N_{irr,f}^4 - N_f^2 N_{irr,f}^2 + \mathcal{J}^2/4 = 0$; taking the larger root, $N_{irr,f}^2 = \frac{1}{2}\big(N_f^2 + \sqrt{N_f^4 - \mathcal{J}^2}\big)$. With the standard definition of the dimensionless spin, and $M_f = N_f m_P$ and $m_P^2 = \hbar c/G$, $\chi_f = Jc/(GM_f^2) = \hbar \mathcal{J} c/(GN_f^2 m_P^2) = \mathcal{J}/N_f^2$; substituting,

$$N_{irr,f}^2 = \frac{N_f^2}{2}\left(1 + \sqrt{1 - \chi_f^2}\right), \qquad \xi_f \equiv \frac{N_{irr,f}}{N_f} = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \chi_f^2}\right)} \tag{50}$$

ξ_f is the irreducible coefficient of the new black hole; in the table, $N_{irr,f} = \xi_f N_f$.

Data: GW150914 from GWTC-1 [32], GW190814 from GWTC-2.1 [33]. For GW190814, rows 7–9 and row 11 are obtained by subtracting observed median values and have large relative errors; they indicate only the order of magnitude. In both cases rows 11 and 12 are equal, a consistency check of the counting in Eq. (24); row 13 does not exceed 1 and row 15 holds, as the area theorem requires. For a very unequal mass ratio (GW190814), the fractions of departing capsules and of relations removed from the irreducible part are both small, and the merger comes closer to incorporation as a whole.

6.3 Unpacking from beginning to end: formation and end of the smallest black hole

Photons per pack. In unpacking $N \to N - 1$, each sub-step has energy $\varepsilon_{N-1} = E_P/(2N-1)$ and entropy $4\pi k_B$, so its temperature is

$$k_B T = \frac{\varepsilon_{N-1}}{4\pi} = \frac{E_P}{4\pi(2N-1)} = k_B T_N^- \tag{51}$$

i.e. the emission temperature of Eq. (15). By Eq. (39), each sub-step is emitted as about $4\pi/2.70 \approx 4.65$ thermal photons. A pack contains $2N - 1$ sub-steps, so the number of photons emitted per pack is

$$n_\gamma = (2N-1) \cdot \frac{4\pi}{2.70} = \frac{E_P}{2.70\, k_B T_N^-} \approx 4.65\,(2N-1) \tag{52}$$

where the second equality uses $4\pi(2N-1) = E_P/(k_B T_N^-)$. For example, a solar-mass black hole with $N = 9.14 \times 10^{37}$ emits about 8.5×10^{38} photons per unpacked pack.

Table 2 Capsules and relations in two mergers

No.	Quantity	Meaning	GW150914	GW190814
1	N_1	Primary black hole	35.6	23.3
2	N_2	Secondary black hole	30.6	2.6
3	N_f	New black hole (incl. rotational energy)	63.1	25.7
4	χ_f	Spin of the new black hole	0.69	0.28
5	ξ_f	Irreducible coefficient, Eq. (50)	0.9284	0.9900
6	$N_{irr,f} = \xi_f N_f$	Irreducible capsule number of the new black hole	58.58	25.44
7	$N_u = N_1 + N_2 - N_{irr,f}$	Capsules leaving the irreducible part	7.62	0.46
8	$N_1 + N_2 - N_f$	Capsules carried off by gravitational waves	3.10	0.20
9	$N_f - N_{irr,f}$	Capsules converted into rotational energy	4.52	0.26
10	N_u/N_2	Fraction of the secondary that leaves	25%	18%
11	$X = (N_1 + N_2)^2 - N_{irr,f}^2$	Relations removed from the irreducible part	950	24
12	$N_u(N_1 + N_2 + N_{irr,f})$	Check: should equal X	950	24
13	$X/(2N_1N_2)$	Fraction of the group relations removed from the irreducible part, should be ≤ 1	44%	20%
14	$N_1 + N_2 - \sqrt{N_1^2 + N_2^2}$	Upper bound on N_u, Eq. (25)	19.26	2.46
15	$N_u \leq$ upper bound	Area-theorem check	Yes	Yes

The smallest black hole is $N = 2$. $N = 1$ is not a black hole, so the formation and the end of a black hole proceed by different processes.

A single E_c neither packs nor unpacks. $N = 1$ is a single E_c; it has no horizon and cannot pack, lacking the conditions for compressing low-energy shares into a pack. Nor can it unpack. By the unpacking rule, unpacking $N \to N - 1$ proceeds in $2N - 1$ sub-steps, each with the share

$$\varepsilon_{N-1} = \frac{E_P}{2N - 1} \tag{53}$$

By the criterion of Eq. (44), a share can be emitted as radiation if

$$\frac{\lambda}{2\pi R_s(\varepsilon_{N-1})} = \frac{1}{2}\left(\frac{E_P}{\varepsilon_{N-1}}\right)^2 = \frac{1}{2}(2N-1)^2 > 1 \tag{54}$$

For $N = 1$, unpacking $1 \to 0$ has only $2 \cdot 1 - 1 = 1$ sub-step, whose share is the whole E_P. Its wavelength is $\lambda = 2\pi\hbar c/E_P = 2\pi\ell_p$ and its own Schwarzschild radius is $R_s(E_P) = 2GE_P/c^4 = 2\ell_p$, so the ratio is

$$\frac{\lambda}{2\pi R_s(E_P)} = \frac{2\pi\ell_p}{2\pi \cdot 2\ell_p} = \frac{1}{2} < 1 \tag{55}$$

This share is still a self-bound pack, i.e. an E_c, and cannot be emitted as radiation; unpacking cannot occur at $N = 1$. For $N = 2$ the share is $E_P/3$ and the ratio is $\frac{1}{2} \cdot 3^2 = 4.5 > 1$, so it can radiate. A ratio greater than 1 requires $(2N-1)^2 > 2$, i.e. $2N - 1 > \sqrt{2}$, whose integer solutions are $N \geq 2$: the smallest N that can unpack is exactly 2, the same boundary as the smallest black hole.

Equation (40) does not apply at $N = 1$ either. Formally, $1 \to 0$ has $\Delta S = -4\pi k_B$ and $\Gamma = e^{-4\pi}$, and with the Planck time as the attempt interval the lifetime of E_c would be only about 10^{-38} s. But Eq. (40) gives the probability of emitting an energy share on the premise that the share can exist as radiation; at $N = 1$ the emitted share is itself self-bound and the premise fails.

The E_c in the cosmic background are therefore stable: it does not decay into radiation, and there is no radiation from E_c in cosmic voids, as is required for E_c to be the unit of dark energy [17].

Formation of the smallest black hole ($E_c + E_c \to N = 2$). A single E_c cannot pack, so $1 \to 2$ cannot be achieved by packing; what enters must be an already complete E_c, i.e. two E_c combine. In the initial state the two E_c each carry a self term; the final state is an $N = 2$ black hole:

$$\frac{\Delta S}{k_B} = 4\pi \cdot 2^2 - (4\pi + 4\pi) = 16\pi - 8\pi = 8\pi \tag{56}$$

i.e. the 2 newly established ordered relations, consistent with Eq. (21) for $N_1 = N_2 = 1$.

For two E_c to combine, the local effective (gravitational) density must reach the capsule contact density. Taking as the two extremes the equivalent cube of a capsule (side $(32\pi/3)^{1/3}\ell_p \approx 3.22\,\ell_p$, with volume equal to the capsule sphere) and the diameter cube (side $4\ell_p$), the contact density is

$$\rho_{\text{contact}} = \frac{m_P}{(4\ell_p)^3} \approx 0.81 \times 10^{95}\ \text{kg/m}^3 \quad \text{to} \quad \frac{m_P}{\frac{4}{3}\pi(2\ell_p)^3} \approx 1.5 \times 10^{95}\ \text{kg/m}^3 \tag{57}$$

The ratio of the two is $\pi/6 \approx 0.52$, the packing fraction of touching spheres in a simple cubic lattice. Such densities were reached only in the hot, dense phase of the early universe [35]; today the effective density of the cosmic background, i.e. the observed dark-energy density, is about 6×10^{-27} kg/m^3, some 10^{121} times below the contact density, while the fluctuations of the ordered cosmic background are only of order 10^{-122}, so the E_c of the cosmic background no longer combine in pairs.

End of the smallest black hole ($N = 2 \to 1$). At $N = 2$ a horizon exists and unpacking is possible. The last step of evaporation is the unpacking $2 \to 1$. By

Eq. (14) with $N = 2$,

$$\Delta S_{-} = -4\pi k_B(2 \cdot 2 - 1) = -12\pi k_B \tag{58}$$

The departing capsule takes away one self term plus 2 ordered relations with the capsule that remains. By Eq. (15), the emission temperature is

$$k_B T_2^{-} = \frac{E_P}{4\pi(2 \cdot 2 - 1)} = \frac{E_P}{12\pi}, \qquad T_2^{-} = \frac{1.417 \times 10^{32}\ \text{K}}{12\pi} \approx 3.76 \times 10^{30}\ \text{K} \tag{59}$$

This is the highest emission temperature of the whole evaporation. The pack is emitted in $2 \cdot 2 - 1 = 3$ sub-steps, each of $E_P/3$; by the hoop criterion of Eq. (44), $\lambda/(2\pi R_s) = \frac{1}{2} \cdot 3^2 = 4.5 > 1$, so the shares are not self-bound and can radiate. By Eq. (52), counted as photons, the pack emits about $3 \times 4.65 \approx 14$ quanta, with mean energy

$$\langle E_\gamma \rangle \approx 2.70\, k_B T_2^{-} = \frac{2.70}{12\pi} E_P \approx 0.072\, E_P \approx 8.7 \times 10^{26}\ \text{eV} \tag{60}$$

These are the most energetic quanta a black hole emits, counted as photons; the actual emission also includes all kinds of massive particles. The ratio of entropy to energy of black-body radiation is $S_{rad} = \frac{4}{3} E/T$ (for a photon gas $S = (E + PV)/T$ with $P = E/3V$; the same holds for any particle species with $k_B T \gg mc^2$), so the entropy of the emitted radiation is

$$S_{rad} = \frac{4}{3} \cdot \frac{E_P}{T_2^{-}} = \frac{4}{3} \cdot 12\pi k_B = 16\pi k_B > |\Delta S_{-}| = 12\pi k_B \tag{61}$$

and the generalized second law holds. The one remaining E_c keeps its self term $4\pi k_B$, which is information intrinsic to E_c; it returns with E_c to the cosmic background and need not be carried by the radiation.

The reverse of the combination of two E_c would be the splitting of an $N = 2$ black hole into two E_c, which "merging without splitting" forbids (Section 4); evaporation can therefore end only with the unpacking $2 \to 1$. A black hole begins with the combination of two E_c and ends with the unpacking $2 \to 1$, leaving one E_c. Last-in-first-out holds for all packing steps ($N \geq 2$); the first step is a combination, not a packing, and is not among them.

6.4 The Page curve

Let a black hole begin evaporating from N_0 packs, and write $S_0 = 4\pi k_B N_0^2$. After k packs have been unpacked, $N = N_0 - k$ packs remain. Write $x = k/N_0$ ($0 \leq x \leq 1$), the fraction of packs unpacked, which is also the fraction of energy emitted. For finite N_0 the final unpacking $2 \to 1$ occurs at $x = 1 - 1/N_0$; the remaining capsule and its self entropy are accounted for separately in paragraph (7), and $x = 1$ in the curves below refers to the macroscopic continuum limit.

(1) **Black-hole entropy.** The black-hole entropy is the coarse-grained entropy of the black hole, i.e. the relational entropy of Section 5. By Eq. (6),

$$\frac{S_{BH}}{S_0} = \frac{(N_0 - k)^2}{N_0^2} = (1 - x)^2 \tag{62}$$

(2) **Outflow entropy and conservation of information.** On unpacking, the relational entropy lost by the black hole (the self term of the departing pack plus its ordered relations with the packs that remain, i.e. $-\Delta S_-$ of Eq. (14)) leaves with the dissolved pack and is carried outside the horizon; we call it the outflow entropy S_{out}. Whatever the black hole loses flows out, with nowhere else to go, so

$$S_{BH} + S_{out} = S_0, \qquad \frac{S_{out}}{S_0} = 1 - (1 - x)^2 = x(2 - x) \tag{63}$$

The relational entropy remaining inside plus that which has flowed out always equals the initial total: black-hole entropy, as information entropy, is conserved. Equation (63) holds exactly at every integer step while the black hole exists ($N \geq 2$); the final step $2 \to 1$ is accounted for in paragraph (7).

Consistency check: summing pack by pack. One can also add up the entropy carried off by each pack without invoking the conservation law. By Eq. (14), the entropy flowing out on unpacking $m \to m - 1$ is $-\Delta S_- = 4\pi k_B(2m - 1)$. Unpacking in turn from N_0 packs, the 1st unpacking has $m = N_0$ and releases $4\pi k_B(2N_0 - 1)$; the 2nd has $m = N_0 - 1$ and releases $4\pi k_B(2N_0 - 3)$; ...; the k-th has $m = N_0 - k + 1$ and releases $4\pi k_B(2N_0 - 2k + 1)$. After k packs have been unpacked, the outflow entropy is the sum of these k terms:

$$S_{out} = 4\pi k_B \left[(2N_0 - 1) + (2N_0 - 3) + \cdots + (2N_0 - 2k + 1)\right]$$

The bracket is an arithmetic series of k terms with first term $2N_0 - 1$ and last term $2N_0 - 2k + 1$; its sum equals the number of terms times the mean of the first and last terms, $k \cdot [(2N_0 - 1) + (2N_0 - 2k + 1)]/2 = k(2N_0 - k)$. Hence

$$S_{out} = 4\pi k_B\, k(2N_0 - k) = 4\pi k_B N_0^2 \cdot \frac{k}{N_0}\left(2 - \frac{k}{N_0}\right) = S_0\, x(2 - x)$$

identical to Eq. (63). The pack-by-pack sum does not assume conservation of information; it uses only the self term and the number of relations that each pack takes away on leaving. The agreement shows that the conservation law, Eq. (63), is exactly the cumulative result of the pack-by-pack outflow.

(3) **Entanglement entropy and the information Page point.** Entanglement entropy is a fine-grained (von Neumann) entropy, measuring the quantum correlations actually present between the radiation and the black hole; coarse-grained entropy counts the possible states when only macroscopic quantities are known and is a capacity. If the black hole and the radiation together are in a pure state (unitary evolution), then by the Schmidt decomposition the entanglement entropies of the radiation and

of the black hole are exactly equal, denoted by the same S_{ent}. Page pointed out that, for typical states, the entanglement entropy of the two parts is approximately the smaller of their two capacities [15]. From the standpoint of information conservation, the information about the black hole carried by the radiation is the outflow entropy, so

$$S_{ent} \approx \min\{S_{out},\, S_{BH}\}, \qquad \frac{S_{ent}}{S_0} = \begin{cases} x(2-x), & 0 \le x \le x_P \\ (1-x)^2, & x_P \le x \le 1 \end{cases} \tag{64}$$

Setting $S_{out} = S_{BH}$:

$$x(2-x) = (1-x)^2 \tag{65}$$

Expanding and rearranging gives $2x^2 - 4x + 1 = 0$, whose root in $0 \le x \le 1$ is

$$x_P = 1 - \frac{1}{\sqrt{2}} \approx 0.293, \qquad \frac{S_{ent}(x_P)}{S_0} = \frac{1}{2} \tag{66}$$

That is, when about 29.3% of the packs have been unpacked, the information of the black hole is exactly half outside and half inside, and the entanglement entropy reaches its maximum $S_0/2$; thereafter the entanglement entropy decreases with further radiation (Fig. 1). This agrees with Page's original result [15], and in the present framework it follows directly from the conservation law, Eq. (63).

Within gravitational theory, the Page curve has recently been obtained from the island formula and replica wormholes [36–39]: after the Page time, part of the black-hole interior (the island) belongs informationally to the radiation. Replica wormholes are Euclidean geometries in the path-integral calculation, not traversable passages, and the physical status of islands in real four-dimensional spacetime is still debated. The present framework does not rely on these constructions: the same curve and the same Page point follow directly from the conservation of relational entropy, Eq. (63), or from the pack-by-pack sum above.

(4) **The thermodynamic Page point.** Another definition of the Page curve takes the coarse-grained entropy of the radiation as the capacity of the radiation side [16]. In real evaporation the black hole radiates into a vacuum at nearly 0 K; heat flows from high to low temperature, an irreversible process that necessarily produces entropy: the field outside the horizon starts in the vacuum (a pure state with zero entropy), the radiation excites a large number of modes, and the coarse-grained entropy of the radiation therefore exceeds the outflow entropy. Only when the black hole exchanges reversibly with radiation in equilibrium at the same temperature are the two equal. Write

$$S_{rad} = \beta S_{out}, \qquad \beta \ge 1 \tag{67}$$

For an ideal black body, unpacking $m \to m-1$ emits E_P at the emission temperature T_m^- (Eq. (15)), with radiated entropy $\frac{4}{3}E_P/T_m^- = \frac{4}{3}\cdot 4\pi k_B(2m-1)$, while the outflow entropy of this pack is $S_{out} = -\Delta S_- = 4\pi k_B(2m-1)$, so $\beta = 4/3$ [40]; including greybody factors, counting only photons and gravitons and radiating into empty vacuum, $\beta \approx 1.48$ [16, 22]. Setting $S_{rad} = S_{BH}$ gives $\beta x(2-x) = (1-x)^2$. Since $x(2-x) = 1-(1-x)^2$, writing $y = (1-x)^2$ gives $\beta(1-y) = y$, $y = \beta/(1+\beta)$, and

therefore

$$x'_P = 1 - \sqrt{\frac{\beta}{1+\beta}} \tag{68}$$

For $\beta = 4/3$, $x'_P \approx 0.244$; for $\beta \approx 1.48$, $x'_P \approx 0.227$.

(5) **Relation between the two Page points.** They answer different questions. The information Page point $x_P \approx 0.293$ answers "when has half of the black hole's information flowed out", depends only on the conservation law Eq. (63) and is independent of β; the thermodynamic Page point x'_P also counts the entropy produced by thermalization outside the horizon as capacity of the radiation, and therefore comes earlier. The extra $(\beta - 1)S_{out}$ from thermalization carries no information about the black hole but measures the irreversibility of evaporation: the total coarse-grained entropy

$$S_{BH} + S_{rad} = S_0 + (\beta - 1)S_{out} \tag{69}$$

increases monotonically during evaporation, and the generalized second law holds. The difference between the two Page points reflects exactly this entropy production; for $\beta = 1$ (reversible) they coincide. From the informational point of view no correction is needed; from the thermodynamic point of view one is. Black-hole thermodynamics and black-hole information part ways here, and are linked again by β.

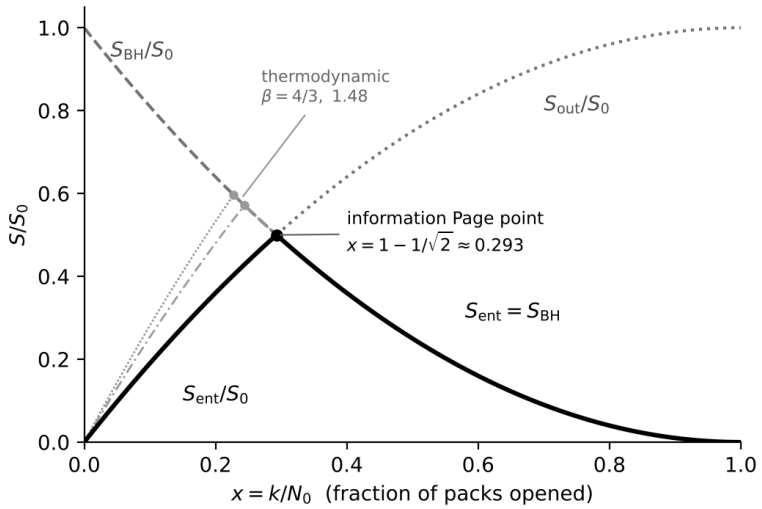

Fig. 1 Page curve of the present framework in the macroscopic continuum limit. The horizontal axis is the fraction of packs unpacked, $x = k/N_0$. The dashed line is S_{BH}/S_0 and the dotted line the outflow entropy S_{out}/S_0 (Eq. (63)); their sum is always 1. The thick solid line is the entanglement entropy (Eq. (64)), with the information Page point $x_P \approx 0.293$. The thin grey lines are the thermodynamic Page curves defined with the coarse-grained radiation entropy, $\min\{\beta S_{out}, S_{BH}\}/S_0$, for $\beta = 4/3$ and $\beta \approx 1.48$, with Page points x'_P of 0.244 and 0.227 respectively (Eq. (68))

(6) **Carriers of the correlations.** The carriers of the outflow entropy are the relations cut across the horizon. In unpacking $m \to m - 1$, the departing pack takes

away its self term and its $2(m-1)$ ordered relations with the $m-1$ packs that remain; it is therefore correlated with each remaining pack. By last-in-first-out, the departing pack is the last to have entered, and those that remain entered before it. Each relation is cut, and counted, only once, when one of its ends is unpacked, so Eq. (63) counts each relation exactly once. How the radiation thermalizes after crossing the horizon does not affect the information accounts here.

(7) **The end point and the content of the information.** After the final unpacking $2 \to 1$, S_{BH} falls from $16\pi k_B$ to zero, leaving one E_c. It has a self term and no relations; and since capsules are intrinsically identical (Section 2), all E_c have the same self term and cannot record any history of the black hole. The remaining E_c is therefore not an information-carrying remnant, and the entanglement entropy must return to zero: all the information is in the radiation. The entropy ledger of this step is $16\pi k_B = 12\pi k_B + 4\pi k_B$: the $12\pi k_B$ released by unpacking consists of one self term and two ordered relations; the remaining E_c is released from the organizing relations of the black hole and returns to the cosmic background, its self entropy $4\pi k_B$ reverting to the intrinsic entropy of a background capsule and no longer counted as black-hole horizon entropy. Over the whole process $S_0 = S_{out} + S_{bg}$, that is, $4\pi k_B N_0^2 = 4\pi k_B(N_0^2 - 1) + 4\pi k_B$, and energy is likewise conserved, $N_0 E_P = (N_0 - 1)E_P + E_P$. Equations (63)–(68) depend only on the number of packs remaining at each step, not on which pack is unpacked, so the Page curve itself does not require last-in-first-out; but the curve gives only the amount of information in the radiation, not its content. Last-in-first-out supplies this: each pack is unpacked at its packing temperature, and the history returns with the radiation in the reverse order of packing.

7 Comparison with other approaches

The coefficient of the area step. Bekenstein and Mukhanov assumed that the horizon area is quantized in equal steps, $A_n = \alpha n \ell_p^2$ [13, 14]. In terms of N, by Eq. (5), $A/\ell_p^2 = 16\pi N^2$, and in the large-N limit an increase of the area by one step $\alpha\ell_p^2$ corresponds to

$$\Delta(16\pi N^2) = 32\pi N\,\Delta N = \alpha, \qquad \Delta N = \frac{\alpha}{32\pi N} \tag{70}$$

The corresponding energy is

$$\Delta E = E_P\,\Delta N = \frac{\alpha}{4}\cdot\frac{E_P}{8\pi N} = \frac{\alpha}{4}\,k_B T_H \tag{71}$$

where the last step uses Eq. (10). They required the degeneracy of the n-th area level to be an integer k^n, so that $\alpha = 4\ln k$ and $\Delta E = \ln k \cdot k_B T_H$; each area step emits one quantum of fixed energy, and the Hawking spectrum becomes a set of equally spaced discrete lines. $k = 2$ is the Bekenstein–Mukhanov value; Hod obtained $k = 3$ from quasinormal modes [41, 42].

The area sub-step of the present framework is $A_o = 16\pi\ell_p^2$, formally $\alpha = 16\pi$, likewise equally spaced in area. There are three differences.

(1) $\alpha = 4\ln k$ gives $k = e^{4\pi} \approx 2.87\times10^5$, which is not an integer, so the Bekenstein–Mukhanov assumption of integer degeneracy is not satisfied. In the present framework 4π is the entropy of one A_o (Eq. (8)), read off from the tunnelling probability $\Gamma_1 = e^{-4\pi}$ (Eq. (41)), and no integer number of microstates is required.

(2) In the present framework energy and area are quantized at two levels: the energy changes in whole steps of E_P, one whole pack containing $2N-1$ area sub-steps; each sub-step carries off an energy $\varepsilon = 4\pi k_B T_N^-$, and only this total energy is fixed. It is emitted as thermal radiation at temperature T_N^-, counted as photons, on average about $4\pi/2.70 \approx 4.65$ photons (Eq. (39)); the energy of each photon follows the Planck distribution, with mean $2.70\,k_B T_N^-$, and the photons are not all alike. A whole pack emits about $4.65(2N-1)$ photons in total (Eq. (52)).

(3) Uniqueness of the step. Let the area unit be A_u, so that the number of units is $n = A/A_u$. If the entropy consists of the self terms and ordered pairwise relations of N' objects, each term carrying entropy σk_B and occupying one area unit, then

$$\begin{aligned} \frac{S}{k_B} &= \sigma\left[P(N',1) + P(N',2)\right] = \sigma\left[N' + N'(N'-1)\right] = \sigma N'^2 \\ n &= \frac{A}{A_u} = N'^2 \end{aligned} \tag{72}$$

Comparing with the Bekenstein–Hawking entropy of Eq. (1) gives

$$\sigma N'^2 = \frac{A_u N'^2}{4\ell_p^2} \;\Rightarrow\; \sigma = \frac{A_u}{4\ell_p^2} \tag{73}$$

From the horizon area $A = 16\pi G^2 M^2/c^4$ and $A = A_u N'^2$,

$$\frac{16\pi G^2 M^2}{c^4} = A_u N'^2 \;\Rightarrow\; M = N'\mu, \qquad \mu = \frac{c^2\sqrt{A_u}}{4\sqrt{\pi}\,G} \tag{74}$$

that is, the combinatorial decomposition requires the black-hole mass to be quantized in equal units μ, and A_u and μ correspond one to one. By the $2d_m$ axiom the energy is quantized in units of E_P, so $\mu c^2 = E_P$, and

$$\sqrt{A_u} = \frac{4\sqrt{\pi}\,G\mu}{c^2} = 4\sqrt{\pi}\,\frac{GE_P}{c^4} = 4\sqrt{\pi}\,\frac{G}{c^4}\sqrt{\frac{\hbar c^5}{G}} = 4\sqrt{\pi}\sqrt{\frac{\hbar G}{c^3}} = 4\sqrt{\pi}\,\ell_p \tag{75}$$

i.e.

$$A_u = 16\pi\ell_p^2 = A_o, \qquad \sigma = \frac{16\pi\ell_p^2}{4\ell_p^2} = 4\pi, \qquad N' = N \tag{76}$$

$\sigma = 4\pi$ agrees with the entropy of one block in Eq. (8) and with the tunnelling probability $\Gamma_1 = e^{-4\pi}$ of Eq. (41). If instead $A_u = 4\ell_p^2$, Eq. (74) gives

$$\mu = \frac{c^2\cdot 2\ell_p}{4\sqrt{\pi}\,G} = \frac{1}{2\sqrt{\pi}}\cdot\frac{E_P}{c^2} \approx 0.282\,\frac{E_P}{c^2} \neq \frac{E_P}{c^2} \tag{77}$$

which is inconsistent with quantization in E_P; without the requirement of a combinatorial decomposition, Eq. (1) is merely a linear sum of identical units and contains no combinatorial structure. Hence A_o is the only area step that yields the combinatorial decomposition of Eq. (7). This derivation uses only the combinatorial rule and quantization in E_P, not the geometric size of the capsule, yet it gives the same A_o and the same entropy 4π per block as obtained earlier from the $2d_m$ axiom, providing an independent check of the capsule size that does not use its geometric dimensions.

The temperature staircase at the end of evaporation. When $k_B T$ exceeds the rest energy $m_e c^2$ of the electron, the lightest charged massive particle, i.e. for $N \lesssim N^* = E_P/(8\pi m_e c^2) \approx 9.5 \times 10^{20}$ ($M \approx 2.1 \times 10^{13}$ kg, $T \approx 5.9 \times 10^9$ K), the emitted particles begin to include massive ones; we call this the end stage of evaporation. Within it, the temperature staircase is significant only for $N \lesssim 10^2$. By Eq. (15), the ratio of the emission temperatures of two successive unpackings is

$$\frac{T^-_{N-1}}{T^-_N} = \frac{E_P/\left[4\pi(2N-3)\right]}{E_P/\left[4\pi(2N-1)\right]} = \frac{2N-1}{2N-3} \tag{78}$$

For large N this is about $1+1/N$: the temperature rises almost continuously, and the thermal radiation of the many sub-steps adds up to a smooth thermal spectrum. For small N the temperature jumps step by step, and the number of sub-steps per pack, $2N-1$, also decreases step by step (Table 3):

Table 3 The last three unpackings at the end of evaporation

Unpacking	$k_B T^-_N$	T^-_N	Sub-steps
$4 \to 3$	$E_P/(28\pi)$	1.61×10^{30} K	7
$3 \to 2$	$E_P/(20\pi)$	2.26×10^{30} K	5
$2 \to 1$	$E_P/(12\pi)$	3.76×10^{30} K	3

The ratios of successive temperatures are 7/5 and 5/3. The end of evaporation is therefore not a continuous process in which the temperature rises smoothly as $1/N$, but a set of discrete thermal peaks: the unpacking $N \to N-1$ emits its $2N-1$ sub-steps with a thermal distribution at temperature T^-_N, the peak positions being proportional to the temperature and discrete as $1/(2N-1)$. The stages appear one after another in time and can be identified one by one in a time-resolved spectrum; in a time-integrated spectrum the width of each thermal peak exceeds the spacing between neighbouring peaks, and they merge into a single broad peak. By last-in-first-out, the packing temperatures fall through the same sequence step by step during accretion and rise back through it in the reverse order during evaporation, in one-to-one correspondence (Fig. 2). This staircase structure is the signature of the present framework at the end of evaporation. It differs in nature from the Bekenstein–Mukhanov lines: within each peak the particle energies still follow the thermal distribution of that stage, and what is discrete is the temperature and total energy of each stage, not the frequency of individual particles. Loop quantum gravity also predicts discrete emission lines at the end of evaporation, lying in the Planck energy range, with evaporation ending

at about $0.4\,m_P$ [43]; the same work notes that the pseudo-periodicity of the area spectrum may leave broad peaks in the instantaneous spectrum of larger black holes, though this is not a necessary prediction of loop quantum gravity [43]. In the present framework the smallest black hole is $N = 2$, i.e. $2m_P$; evaporation ends with the unpacking $2 \to 1$, leaving one E_c, and for large N the spectrum is smooth and reduces to the Hawking thermal spectrum. The staircase is significant only for N below about 10^2: there $k_B T_N^-$ exceeds 5×10^{24} eV, far above the rest energies of all known particles (about 1.7×10^{11} eV for the top quark), and more than four orders of magnitude above the highest energies now detected (about 10^{15} eV for gamma rays and 3×10^{20} eV for cosmic rays), and this stage lasts about 10^{-33} s. Conversely, bringing the particle energies into the detectable range requires $N \gtrsim 10^6$, where the temperature difference between adjacent packs is about $1/N \lesssim 10^{-6}$, indistinguishable from a continuous thermal spectrum. At the end of evaporation the energy of each sub-step, $\varepsilon_{N-1} = E_P/(2N-1)$, is shared by only a few particles, and if conserved quantum numbers are balanced by particle–antiparticle pairs within a sub-step and the sub-step releases a single pair, each particle has energy $E_P/[2(2N-1)]$, so the spectrum at the end may be closer to discrete lines than a thermal peak, its detailed shape depending on the particle species, which is not pursued here.

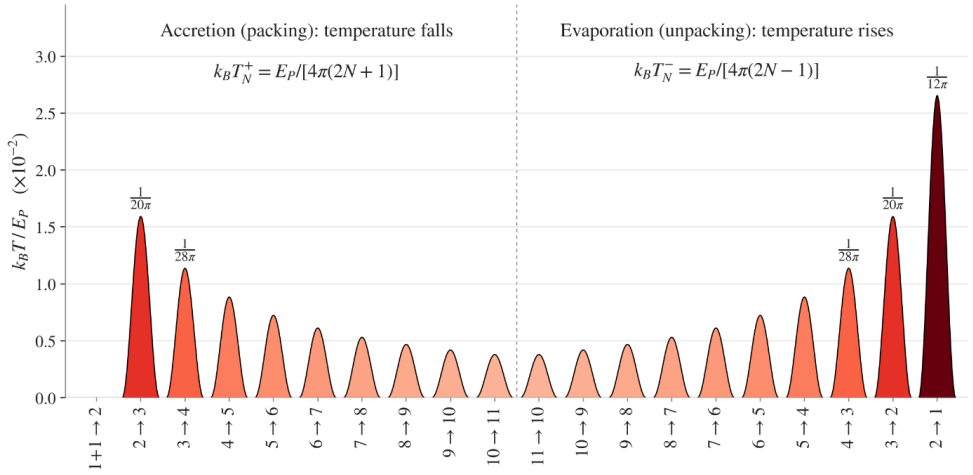

Fig. 2 Temperatures of the successive steps over the life of a black hole. Left: accretion (packing $N \to N+1$), $k_B T_N^+ = E_P/[4\pi(2N+1)]$, falling as N grows. Right: evaporation (unpacking $N \to N-1$), $k_B T_N^- = E_P/[4\pi(2N-1)]$, rising as N falls. Peaks of the same colour have equal temperatures, $T_N^+ = T_{N+1}^-$ (Eq. (46)), i.e. last in, first out. The first step $1+1 \to 2$ is the combination of two E_c, absorbs no energy and has no temperature; the last step $2 \to 1$ is the hottest. Peak shapes are schematic; peak heights give the step temperature

The coefficient of the logarithmic correction. Loop quantum gravity gives a correction of $-\frac{3}{2}\ln A$ or $-\frac{1}{2}\ln A$, depending on the counting scheme [44, 45], and some work argues that large black holes have no logarithmic correction [46]. Sen derived the

logarithmic correction for Schwarzschild black holes from one-loop (infrared) effects of massless fields, with a coefficient that depends on the massless field content [47]. In the present framework the relational entropy $4\pi N^2$ of the horizon is exact, and the relational counting itself contains no logarithmic term.

8 Entropy of Kerr black holes

8.1 Irreducible mass and the two horizons

For a Kerr black hole of mass M and angular momentum J, write $a \equiv J/(Mc)$; the two horizons are $r_\pm = GM/c^2 \pm \sqrt{(GM/c^2)^2 - a^2}$ and the horizon area is $A = 8\pi GMr_+/c^2$ [48]. The outer horizon satisfies $r_+ \leq 2GM/c^2$, with equality only for $a = 0$, so the correspondence of Section 2 cannot be applied directly. Christodoulou defined the irreducible mass as the mass of the Schwarzschild black hole with horizon area A [34]:

$$M_{irr} \equiv \frac{c^2}{G}\sqrt{\frac{A}{16\pi}} \tag{79}$$

The irreducible mass and the total mass are related by

$$M^2 = M_{irr}^2 + \frac{J^2c^2}{4G^2M_{irr}^2} \tag{80}$$

M_{irr} cannot be decreased by any classical process. For the extremal Kerr black hole, $a = GM/c^2$ and $J = aMc = GM^2/c$; substituting into Eq. (80) gives $M^2 = M_{irr}^2 + M^4/(4M_{irr}^2)$, i.e. $M_{irr}^4 - M^2M_{irr}^2 + M^4/4 = (M_{irr}^2 - M^2/2)^2 = 0$, so that $M_{irr} = M/\sqrt{2}$, and the extractable rotational energy is at most $(1 - 1/\sqrt{2})Mc^2 \approx 29.3\%\, Mc^2$ [34, 49].

The two horizons of the Kerr solution are

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2} \tag{81}$$

from which follow directly

$$r_+ + r_- = \frac{2GM}{c^2} = R_s, \qquad r_+r_- = a^2 \tag{82}$$

It should be stressed that $r_\pm$ are the coordinate values at which the horizons lie, not radii of spheres; between the two horizons r is timelike, and the coordinate difference is not a spatial distance. The horizon of a Kerr black hole is an oblate surface: rotation makes it bulge at the equator and flatten at the poles. The true size of a horizon is given by its area. On a horizon ($\Delta = r^2 - R_sr + a^2 = 0$), the angular part of the metric is $g_{\theta\theta} = r_\pm^2 + a^2\cos^2\theta$ and $g_{\phi\phi} = (r_\pm^2 + a^2)^2\sin^2\theta/(r_\pm^2 + a^2\cos^2\theta)$, whose product is

$(r_\pm^2 + a^2)^2 \sin^2\theta$; integrating gives the area

$$A_\pm = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \left(r_\pm^2 + a^2\right) \sin\theta \, \mathrm{d}\theta = 4\pi \left(r_\pm^2 + a^2\right) = 4\pi R_s r_\pm \tag{83}$$

The last step uses $a^2 = r_+ r_-$: $r_\pm^2 + a^2 = r_\pm(r_+ + r_-) = R_s r_\pm$. Defining the radius of a horizon by its area, $R_{A\pm} = \sqrt{A_\pm/(4\pi)} = \sqrt{r_\pm^2 + a^2} = \sqrt{R_s r_\pm}$.

With the dimensionless spin $\chi \equiv ac^2/(GM)$, $r_\pm = (GM/c^2)\big(1 \pm \sqrt{1-\chi^2}\big)$, and the areal radii can be written as

$$R_{A\pm} = \sqrt{R_s r_\pm} = \frac{R_s}{2}\left(\sqrt{1+\chi} \pm \sqrt{1-\chi}\right) \tag{84}$$

The moment of inertia and the other aspects of the non-spherical geometry of Kerr black holes are not discussed in this paper.

8.2 Distribution of entropy over the two horizons

Entropy: direct substitution. By definition, a Kerr black hole has the same horizon area as a Schwarzschild black hole of mass M_{irr}. For every result that depends only on the horizon area, it suffices to replace N in Sections 2–6 by $N_{irr} \equiv M_{irr}/m_P$:

$$\frac{S_{BH}}{k_B} = 4\pi N_{irr}^2 = 4\pi\left[N_{irr} + N_{irr}(N_{irr} - 1)\right] \tag{85}$$

By Eq. (1), the entropy of each horizon is $S_\pm = k_B A_\pm/(4\ell_p^2)$, so that

$$S_\pm = \frac{\pi k_B R_s r_\pm}{\ell_p^2}, \qquad S_{tot} = S_+ + S_- = \frac{\pi k_B R_s (r_+ + r_-)}{\ell_p^2} = \frac{\pi k_B (2N\ell_p)^2}{\ell_p^2} = 4\pi k_B N^2 \tag{86}$$

where $N = M/m_P$ counts the total mass. The entropy of the outer horizon can be derived from Eqs. (79) and (83). By Eq. (79), $M_{irr}^2 = c^4 A_+/(16\pi G^2)$; substituting $A_+ = 4\pi R_s r_+$,

$$M_{irr}^2 = \frac{c^4 \cdot 4\pi R_s r_+}{16\pi G^2} = \frac{c^4 R_s r_+}{4G^2}, \qquad N_{irr}^2 = \frac{M_{irr}^2}{m_P^2} = \frac{c^4 R_s r_+}{4G^2 m_P^2} = \frac{R_s r_+}{4\ell_p^2} \tag{87}$$

The last step uses $Gm_P/c^2 = \ell_p$, i.e. $G^2 m_P^2/c^4 = \ell_p^2$. Comparing with Eq. (86),

$$S_+ = \frac{\pi k_B R_s r_+}{\ell_p^2} = 4\pi k_B \cdot \frac{R_s r_+}{4\ell_p^2} = 4\pi k_B N_{irr}^2 \tag{88}$$

The entropy of the inner horizon follows independently from its own area (counting entropy by the area of the inner horizon and giving it a physical meaning has been

discussed before [50]). By Eq. (82), $r_- = R_s - r_+$; substituting into Eq. (86),

$$S_- = \frac{\pi k_B R_s r_-}{\ell_p^2} = \frac{\pi k_B R_s (R_s - r_+)}{\ell_p^2} = \frac{\pi k_B R_s^2}{\ell_p^2} - \frac{\pi k_B R_s r_+}{\ell_p^2} = 4\pi k_B N^2 - 4\pi k_B N_{irr}^2 \tag{89}$$

In the last step the first term uses $R_s = 2N\ell_p$ and the second Eq. (87). The entropies of the two horizons are each obtained from their own areas, and their sum is

$$S_{tot} = S_+ + S_- = 4\pi k_B N_{irr}^2 + 4\pi k_B \left(N^2 - N_{irr}^2\right) = 4\pi k_B N^2 \tag{90}$$

which is exactly the entropy counted by the total energy, in agreement with Eq. (86). The outer horizon carries exactly the irreducible part, in agreement with Eq. (85); the inner horizon carries the rest, i.e. the part due to rotation. Geometrically, the sum of the two horizon areas is

$$A_+ + A_- = 4\pi R_s (r_+ + r_-) = 4\pi R_s^2 \tag{91}$$

i.e. the sum of the two horizon areas is always $4\pi R_s^2$, fixed by the mass alone and independent of spin. The equatorial circumference of both horizons is always $2\pi\sqrt{g_{\phi\phi}}\,\big|_{\theta=\pi/2} = 2\pi(r_\pm^2 + a^2)/r_\pm = 2\pi R_s$, likewise independent of spin. Compare two Kerr black holes of the same mass and different spin: the one with the larger spin has smaller r_+ and larger r_-, hence a smaller outer-horizon area $A_+ = 4\pi R_s r_+$ and a larger inner-horizon area $A_- = 4\pi R_s r_-$, their sum being unchanged. This is a comparison between two black holes, not a process: in classical accretion, at fixed mass the spin cannot increase. By the first law of black-hole mechanics and the area theorem, adding angular momentum requires adding energy, $\mathrm{d}Mc^2 \geq \Omega_H\,\mathrm{d}J$ [3]; as the spin increases, M increases and the sum of the two horizon areas grows; how each horizon changes depends on the type of pack absorbed (after Eq. (92)), the share of the inner horizon increasing with spin (Eq. (98)).

Position. The usual Kerr black-hole entropy is the outer-horizon entropy S_+; the present framework also counts the entropy corresponding to the inner-horizon area. The framework interprets this as follows: all capsules of the black hole are packed, and the total number of relations $4\pi N^2$ is shared between the two horizons in proportion to their areas (Eq. (90)). Writing $N_{rot} \equiv N - N_{irr}$, the shares of the two horizons are

$$\begin{aligned} S_+/k_B &= 4\pi N_{irr}^2 = 4\pi N_{irr} + 4\pi N_{irr}(N_{irr} - 1) \\ S_-/k_B &= 4\pi\left(N^2 - N_{irr}^2\right) = 4\pi N_{rot}^2 + 8\pi N_{irr} N_{rot} \end{aligned} \tag{92}$$

By Eq. (92), for the two elementary accretion routes: absorbing one irreducible pack increases the outer-horizon entropy by $4\pi k_B(2N_{irr}+1)$ and the inner-horizon entropy, through cross relations, by $8\pi k_B N_{rot}$ (unchanged when $N_{rot} = 0$); absorbing one rotational pack leaves the outer-horizon entropy unchanged and increases the inner-horizon entropy by $4\pi k_B(2N+1)$. The total entropy increase is the same for both routes, $4\pi k_B(2N+1)$, the entropy change of one pack; they differ only in how the new

relations are distributed between the two horizons. The outer-horizon area therefore never decreases, consistent with the area theorem.

The outer horizon registers the self terms and mutual relations of the irreducible part; the inner horizon registers the self terms and mutual relations of the rotational part, plus its cross relations with the irreducible part, with the same structure as the entropy increase when two parts combine (Section 4.2). The two horizons only register; packing and unpacking both take place at the inner interface $R_i = R_s/2$ (R_i taken as the areal radius of the packing interface). The areal radius of the inner horizon, $R_{A-} = R_s \sin\omega$ (Section 8.3), starts from zero and coincides with R_i at $\omega = \pi/6$ ($\chi = \sqrt{3}/2$). This crossing requires $N_{irr}/N = \cos(\pi/6) = \sqrt{3}/2$, which is irrational, so it cannot be realized exactly by finite integers N and N_{irr}, for the same reason as the extremal state; beyond this the inner horizon lies outside R_i, and packs still enter R_i to be packed. Rotational energy can be extracted in the ergosphere through the Penrose process [51]; what is extracted is the rotational part, and N_{irr} does not decrease, consistent with the irreducible mass not being decreasable by classical processes.

8.3 Trigonometric geometry of the entropy distribution

We first write Eq. (80) in terms of capsule numbers. Write $\mathcal{J} \equiv J/\hbar$, with $N = Mc^2/E_P$ as in Eq. (86). Multiplying both sides of Eq. (80) by c^4/E_P^2 and using $E_P^2 = \hbar c^5/G$:

$$\frac{J^2c^6}{4G^2M_{irr}^2E_P^2} = \frac{\hbar^2\mathcal{J}^2c^6}{4G^2E_P^2}\cdot\frac{c^4}{N_{irr}^2E_P^2} = \frac{\mathcal{J}^2}{4N_{irr}^2}\cdot\frac{\hbar^2c^{10}}{G^2E_P^4} = \frac{\mathcal{J}^2}{4N_{irr}^2} \tag{93}$$

so that

$$N^2 = N_{irr}^2 + \frac{\mathcal{J}^2}{4N_{irr}^2} \tag{94}$$

The distribution over the two horizons can be expressed by a single angle. Let $N_{irr} = N\cos\omega$; then $\sqrt{N^2 - N_{irr}^2} = N\sin\omega$. By Eq. (87), using $R_s = 2N\ell_p$ and $r_+ = (GM/c^2)\big(1+\sqrt{1-\chi^2}\big) = N\ell_p\big(1+\sqrt{1-\chi^2}\big)$,

$$N_{irr}^2 = \frac{R_s r_+}{4\ell_p^2} = \frac{2N\ell_p\cdot N\ell_p\left(1+\sqrt{1-\chi^2}\right)}{4\ell_p^2} = \frac{N^2\left(1+\sqrt{1-\chi^2}\right)}{2} \tag{95}$$

Comparing with $N_{irr}^2 = N^2\cos^2\omega$ gives $\cos^2\omega = \big(1+\sqrt{1-\chi^2}\big)/2$, and by the double-angle formula

$$\cos 2\omega = 2\cos^2\omega - 1 = \sqrt{1-\chi^2}, \qquad \sin 2\omega = \sqrt{1-\cos^2 2\omega} = \chi \tag{96}$$

i.e. 2ω is the spin angle, $0 \le \omega \le \pi/4$. By Eq. (94),

$$\mathcal{J} = 2N_{irr}\sqrt{N^2 - N_{irr}^2} = 2N\cos\omega\cdot N\sin\omega = N^2\sin 2\omega = \chi N^2 \tag{97}$$

in agreement with general relativity: $\mathcal{J} = J/\hbar = aMc/\hbar = \chi GM^2/(\hbar c) = \chi M^2/m_P^2 = \chi N^2$. Now consider the two horizons. By Eq. (81), $r_\pm = GM/c^2 \pm \sqrt{(GM/c^2)^2 - a^2}$; using $a = \chi GM/c^2$ gives $r_\pm = GM/c^2 \pm \sqrt{(GM/c^2)^2 - \chi^2(GM/c^2)^2} = (GM/c^2)\big(1 \pm \sqrt{1-\chi^2}\big)$, and by Eq. (96), $\sqrt{1-\chi^2} = \cos 2\omega$; with $GM/c^2 = R_s/2$, $r_\pm = (R_s/2)(1 \pm \cos 2\omega)$. Substituting into Eq. (84) and comparing with Eqs. (88) and (89),

$$\begin{aligned} R_{A+} &= \sqrt{R_s \cdot \frac{R_s}{2}\,(1+\cos 2\omega)} = R_s \cos\omega, \qquad & S_+ &= 4\pi k_B N^2 \cos^2\omega \\ R_{A-} &= \sqrt{R_s \cdot \frac{R_s}{2}\,(1-\cos 2\omega)} = R_s \sin\omega, \qquad & S_- &= 4\pi k_B N^2 \sin^2\omega \end{aligned} \tag{98}$$

The areal radii of the two horizons are exactly the two legs of a right triangle with hypotenuse R_s: by Eq. (91), the three radii satisfy the Pythagorean theorem $R_{A+}^2 + R_{A-}^2 = R_s^2$, and the entropy sum of Eq. (90) is $\cos^2\omega + \sin^2\omega = 1$. This triangle has a direct geometric origin: the horizon condition $\Delta = 0$ can be written as $(r - R_s/2)^2 + a^2 = (R_s/2)^2$, which in the (r, a) plane is a circle of radius $R_s/2$ and diameter R_s; the point $P(r_+, a)$ lies on the circle, and by Thales' theorem the angle subtended at P by the diameter OB is a right angle, the angle at O is ω, and the central angle of P is 2ω (Fig. 3). Since $\cos\omega = M_{irr}/M$, Eq. (95) is the geometric form of Christodoulou's irreducible-mass relation [34]. Without rotation, $\omega = 0$ and $R_{A-} = 0$; in the extremal case, $\omega = \pi/4$ and $R_{A+} = R_{A-} = R_s/\sqrt{2}$, in agreement with Section 8.1.

Writing the fraction of rotational capsules as $x \equiv N_{rot}/N = 1 - \cos\omega$, Eq. (98) gives $S_+/S_{tot} = \cos^2\omega = (1-x)^2$ and $S_-/S_{tot} = \sin^2\omega = 1 - (1-x)^2 = x(2-x)$, the same form as the Page curve, Eqs. (62) and (63): the rotational capsule number N_{rot} corresponds to the packs that can be unpacked, the outer horizon to the black-hole interior, and the inner horizon to the radiation. In the extremal case $S_+ = S_-$: the entropies of the two horizons balance. With $S_+/S_{tot} = \cos^2\omega = (1-x)^2$ and $S_-/S_{tot} = \sin^2\omega = x(2-x)$, the condition $S_+ = S_-$ gives

$$(1-x)^2 = x(2-x) \tag{99}$$

Expanding and collecting terms, $1 - 2x + x^2 = 2x - x^2$, that is $2x^2 - 4x + 1 = 0$, with roots $x = 1 \pm \frac{1}{\sqrt{2}}$. Since $0 \le x \le 1$, only the root $x = 1 - \frac{1}{\sqrt{2}}$ is physical, and

$$\frac{N_{rot}}{N} \equiv x = 1 - \frac{1}{\sqrt{2}} = 0.293, \qquad \frac{N_{irr}}{N} \equiv 1 - x = \frac{1}{\sqrt{2}} = 0.707 \tag{100}$$

This balance point has the same value as the information Page point x_P of Eq. (66). The maximum extractable rotational energy of 29.3% stated in Section 8.1 is exactly this balance point: the extractable rotational energy can at most equalize the entropies of the two horizons, and the outer horizon always carries more than half of the entropy. By the existence criterion of Section 8.4, the extremal state is not an allowed integer state, so 29.3% is an upper bound in the continuum limit, not reached by black holes of finite N.

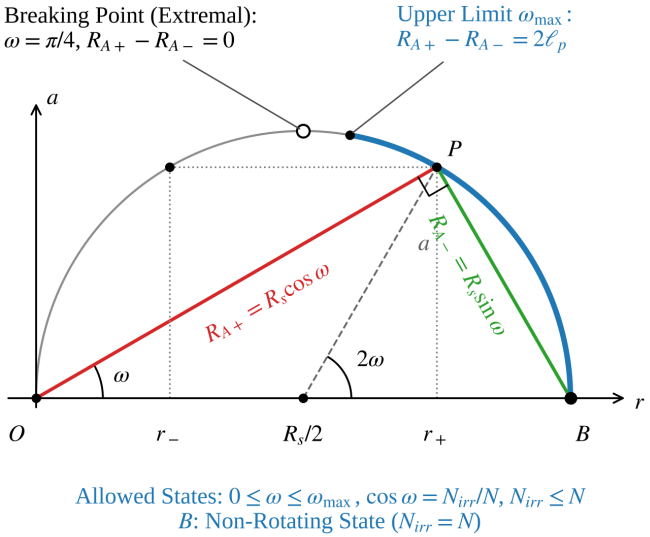

Fig. 3 Horizon circle. In the (r, a) plane the horizon condition $\Delta = 0$ is a circle centred at $(R_s/2, 0)$ with radius $R_s/2$ and diameter $OB = R_s$. A point P on the circle has central angle 2ω, with $\sin 2\omega = \chi$, and inscribed angle ω; the triangle OPB has a right angle at P, and its legs are the areal radii of the two horizons, $R_{A+} = R_s \cos\omega$ and $R_{A-} = R_s \sin\omega$. The thick arc is the allowed range $0 \le \omega \le \omega_{\max}$, the upper limit $\omega_{\max}$ being given by $R_{A+} - R_{A-} = 2\ell_p$ (ℓ_p exaggerated); the allowed states are the discrete points $\cos\omega = N_{irr}/N$ ($N_{irr} \le N$) on it; the open circle at $\omega = \pi/4$ is the breaking point of the extremal state; the filled point B is the non-rotating state $N_{irr} = N$

8.4 Existence criterion for black holes

The two horizons cannot coincide: the difference of their areal radii, $R_{A+} - R_{A-}$, cannot be smaller than $2\ell_p$. Each horizon is an independent spacetime-geometric interface requiring a minimal thickness ℓ_p, and a gap of ℓ_p is needed between them, so the difference of the areal radii of the two interfaces is at least $2\ell_p$. The two horizons register the irreducible and the rotational parts respectively, and if they coincided, the information registered on the two surfaces would mix and could no longer be distinguished:

$$R_{A+} - R_{A-} = R_s\sqrt{1-\chi} \;\ge\; 2\ell_p \tag{101}$$

In the extremal case $\chi = 1$, $R_{A\pm} = R_s/\sqrt{2}$ and $R_{A+} - R_{A-} = 0$: this is the breaking point of the extremal Kerr state. $R_{A+} - R_{A-} = 2\ell_p$ is the lower limit of the separation. This is consistent with the third law of black-hole thermodynamics [52] (a black hole cannot reach the extremal state). For $\chi = 0$, $R_{A+} = R_s$ and $R_{A-} = 0$, the Schwarzschild case; the inner horizon itself has no lower limit. The constraint Eq. (101) becomes $R_s(\cos\omega - \sin\omega) \ge 2\ell_p$. For $\omega < \pi/4$ both sides are positive and may be squared; using $(\cos\omega - \sin\omega)^2 = \cos^2\omega + \sin^2\omega - 2\sin\omega\cos\omega = 1 - \sin 2\omega$ and

$2\ell_p/R_s = 1/N$, we obtain the criterion for the spin angle

$$\sin 2\omega \le 1 - \frac{1}{N^2}, \qquad \omega \le \omega_{\max} = \frac{1}{2}\arcsin\left(1 - \frac{1}{N^2}\right) < \frac{\pi}{4} \tag{102}$$

i.e. $\chi \le 1 - 1/N^2$, in agreement with Eq. (101); $\omega_{\max}$ is the end point of the thick arc in Fig. 3 and is the upper limit; the top of the circle ($\omega = \pi/4$) is the breaking point of the extremal Kerr state.

Eq. (102) constrains the extremal side; on the other side only $N_{irr} \le N$ is required,

$$\frac{\sqrt{2}}{2} < \frac{N_{irr}}{N} \le 1 \tag{103}$$

The left end, $N_{irr}/N = \sqrt{2}/2$, is the extremal state $M_{irr} = M/\sqrt{2}$ of Section 8.1, and the right end, $N_{irr} = N$, does not rotate; N_{irr} is an integer, and the allowed states are the discrete points $\cos\omega = N_{irr}/N$ on the allowed arc of Fig. 3. $N = 1$ is the E_c capsule, not a black hole. For $N = 2$ and 3 the only N_{irr} in this range is $N_{irr} = N$; for $N = 4$, $N_{irr} = 3$ violates Eq. (102); black holes with $N = 2$–4 therefore do not rotate. The smallest rotating state is $N = 5$, $N_{irr} = 4$, $\chi = 0.96$, the separation of its two horizons being exactly $2\ell_p$. With $R_{A+} = 2\ell_p N_{irr}$ and $R_{A-} = 2\ell_p\sqrt{N^2 - N_{irr}^2}$, Eq. (101) reads $N_{irr} - \sqrt{N^2 - N_{irr}^2} \ge 1$. For $N_{irr} = N - 1$ it becomes $N - 2 \ge \sqrt{2N-1}$, i.e. $(N-1)(N-5) \ge 0$: every $N \ge 5$ has a rotating state satisfying the existence criterion, with equality at $N = 5$, i.e. the $(5, 4)$ state whose separation is exactly $2\ell_p$.

From the counting side, N and N_{irr} are both capsule numbers and take integer values, so that $\mathcal{J}$ takes the discrete values given by Eq. (97), yielding a discrete angular-momentum spectrum. The extremal Kerr black hole requires $\omega = \pi/4$, i.e. $N = \sqrt{2}\,N_{irr}$, with equal shares of the two horizons, $S_+ = S_-$; since $\sqrt{2}$ is irrational, no integers N and N_{irr} satisfy this, so in the counting the extremal state does not correspond to any black-hole state either. This shows from the other side that Eq. (101) is reasonable: geometrically, the difference of the areal radii of the two horizons cannot vanish; in the counting, the shares of the two horizons cannot be equal. From $N_{irr} - \sqrt{N^2 - N_{irr}^2} \ge 1$, rearranged as $2N_{irr}^2 - 2N_{irr} + 1 - N^2 \ge 0$, the allowed integer range is $\lceil (1 + \sqrt{2N^2 - 1})/2 \rceil \le N_{irr} \le N$: for $N \le 4$ only $N_{irr} = N$, and for $N = 5$, $N_{irr} = 4$ or 5.

8.5 Kerr temperature

Temperature: no direct substitution. Substituting N_{irr} directly into Eq. (10) gives the temperature of a Schwarzschild black hole of mass M_{irr}, which does not hold for a Kerr black hole: the extremal Kerr black hole has a temperature of 0 K, whereas $E_P/(8\pi N_{irr})$ is finite. The reason is that $E = Mc^2$ in the temperature $T = \partial E/\partial S$ (at fixed J) is the total energy, including rotational energy, while N_{irr} counts only the irreducible part.

In the continuum thermodynamic limit, differentiating with respect to N_{irr} at fixed $\mathcal{J}$: $2N\,\partial N/\partial N_{irr} = 2N_{irr} - \mathcal{J}^2/(2N_{irr}^3)$; and by Eq. (94) again, $\mathcal{J}^2/(4N_{irr}^2) =$

$N^2 - N_{irr}^2$, so

$$\frac{\partial E}{\partial N_{irr}} = E_P \frac{\partial N}{\partial N_{irr}} = E_P \, \frac{N_{irr}^2 - (N^2 - N_{irr}^2)}{N_{irr}\, N} = E_P \, \frac{2N_{irr}^2 - N^2}{N_{irr}\, N} \tag{104}$$

By Eq. (85), $\mathrm{d}S_{BH}/\mathrm{d}N_{irr} = 8\pi k_B N_{irr}$, hence

$$k_B T = \frac{E_P}{8\pi N_{irr}} \cdot \frac{2N_{irr}^2 - N^2}{N_{irr}\, N} \tag{105}$$

The first factor is the temperature of the equivalent Schwarzschild black hole, the second the correction due to rotation. Its meaning is that when N_{irr} increases by 1 at fixed J, the total energy increases by less than E_P, because as the irreducible part grows the rotational energy falls, offsetting part of the increase. For $a = 0$, $N = N_{irr}$, the correction factor is 1 and Eq. (105) reduces to Eq. (10); in the extremal case $N^2 = 2N_{irr}^2$, the correction factor is 0 and the temperature is 0 K. Substituting $N_{irr}^2 = c^2 M r_+/(2Gm_P^2)$ (from $A = 8\pi GMr_+/c^2$ and Eq. (79)) and $N = M/m_P$, Eq. (105) becomes $k_B T = \hbar c^3 \sqrt{(GM/c^2)^2 - a^2}/(4\pi GMr_+)$, which is exactly the Hawking temperature of a Kerr black hole, $\hbar\kappa/(2\pi c)$ [2]. Substituting $N_{irr} = N\cos\omega$ into Eq. (105), the second factor becomes $(2N^2\cos^2\omega - N^2)/(N\cos\omega \cdot N) = (2\cos^2\omega - 1)/\cos\omega = \cos 2\omega/\cos\omega$ and the first $E_P/(8\pi N\cos\omega)$, so that

$$k_B T = \frac{E_P}{8\pi N} \cdot \frac{\cos 2\omega}{\cos^2\omega} = \frac{E_P}{8\pi N}\left(1 - \tan^2\omega\right) \tag{106}$$

which differs from Eq. (10) only by a trigonometric factor. By Fig. 3, $\tan\omega = a/r_+ = R_{A-}/R_{A+}$, so $1 - \tan^2\omega = (A_+ - A_-)/A_+$: the rotational correction to the temperature is the difference of the two horizon areas as a fraction of the outer-horizon area. The temperature vanishes when the two horizon areas are equal, which is exactly the breaking point of Section 8.4; before breaking, the temperature of a black hole does not fall to 0 K. The entropy counting combined with the energy function Eq. (80) reproduces Kerr thermodynamics, but Eq. (80) is taken from general relativity and is not derived independently by the framework.

8.6 Stability of the Cauchy horizon

In classical general relativity, infalling radiation is infinitely blueshifted near the Cauchy horizon [53], causing mass inflation [54], so that the Cauchy horizon was long regarded as classically unstable, with diverging curvature; with a positive cosmological constant it may be classically stable [55], but the quantum stress tensor diverges there [56]. More recent rigorous results show that the Cauchy horizon of a Kerr black hole is stable in the C^0 sense [57]: the metric extends continuously, and the inner horizon does not disappear as a surface. In the present framework, the area of the outer horizon changes in sub-steps of A_o, and packing and unpacking at the inner interface R_i proceed in whole packs of E_P; the inner horizon only registers relations and holds no packed energy. Energy can cross the outer horizon only in sub-steps and

is packed at R_i; the amount packed equals the amount that crossed the outer horizon, and there is no continuous blueshift amplification. The push on R_i is exactly the maximum force, $E/(4R_i) = c^4/(4G)$, so the energy within R_i is always $E = c^4 R_i/G$; at fixed R_i no further energy can accumulate, since the push would then exceed the maximum force [17], and each pack E_P moves R_i outward by $\Delta R_i = GE_P/c^4 = \ell_p$, with R_s moving outward by $2\ell_p$. For spin $\chi < \sqrt{3}/2$ the inner horizon lies inside R_i, in the capsule region: accreted shares are packed at R_i and unpacked shares tunnel outward from R_i, so there is no energy flux inside R_i; the capsule environment is transparent from within, offering no resistance, like the E_c of the cosmic background, and the crossing fluxes required for mass inflation are absent. For $\chi > \sqrt{3}/2$ the inner horizon enters the pure-spacetime region between R_i and the outer horizon. First, by the $2d_m$ axiom no elementary quantum exceeds E_P ($\lambda_{\min} = 2\pi\ell_p$) [17]; by the hoop criterion, a share whose local energy exceeds $E_P/\sqrt{2} \approx 0.707E_P$ is self-bound and can no longer propagate as free radiation (Eq. (44)). Blueshift amplification is therefore cut off before reaching E_P, leaving only small, finite fluctuations, and the local energy density does not diverge. Packing takes place only at R_i. Second, within a black hole a rotational pack can be re-registered as an irreducible pack but not the reverse, which would decrease the outer-horizon area in violation of the area theorem; at fixed N, re-registration lowers χ and the inner horizon retreats, the angular momentum released being carried off by radiation or the external field. Together these suppress fluctuations near the inner horizon and provide a mechanism for suppressing mass inflation.

9 Conclusions

Taking the Bekenstein–Hawking entropy formula $S_{BH} = k_B A/(4\ell_p^2)$ as the basis, combined with the $2d_m$ axiom established in our previous paper, and starting from the entropy decomposition of Eq. (7), $S_{BH}/k_B = 4\pi N^2 = 4\pi N + 4\pi N(N-1) = 4\pi[P(N,1) + P(N,2)]$, we have obtained the following results, which show content that the Bekenstein–Hawking formula cannot display.

Quantized accretion and evaporation.

1. Counting by the number N of E_c capsules reproduces the Hawking temperature, the heat capacity $C = -2S$ and the Smarr relation. It reveals that a relation between E_c capsules has the weight 4π, the same as a capsule self term, and excludes three-body terms. Capsules are intrinsically identical but distinguishable inside a black hole by the order of packing, so they are counted by permutations.
2. Integer N gives discrete absorption and emission temperatures $T_N^{\pm} = T_H \cdot 2N/(2N \pm 1)$, whose harmonic mean is the Hawking temperature.
3. The entropy is superadditive: when N_1 and N_2 capsules combine, the horizon entropy increase $8\pi k_B N_1 N_2$ means that the cross relations between the capsules of the two parts are counted in the horizon; when two black holes merge, these relations are converted from the group relations between the two before the merger; no sub-black-hole survives a merger, and the entropy depends only on the total capsule number, which is a counting statement of the no-hair theorem. What is additive is $\sqrt{S}$, $S = 4\pi k_B(\sum_i N_i)^2 = (\sum_i \sqrt{S_i})^2$; the relations removed from the irreducible part in a merger by the capsules leaving it (gravitational waves and

rotational energy) do not exceed the $2N_1N_2$ group relations, and the corresponding bound is Hawking's bound. The area law cannot come from counting states in the volume. "Most disordered" and "most ordered" are the external and internal readings of the same S_{BH}.

4. Accretion and evaporation are quantized at two levels: energy in whole packs of E_P, area in sub-steps of A_o. Each sub-step has tunnelling probability $e^{-4\pi}$, independent of N, while the probability of emitting a whole pack at once is suppressed to $e^{-4\pi(2N-1)}$; each sub-step is emitted as about 4.65 thermal photons on average, and one whole pack E_P is divided into $2N-1$ sub-steps. The hoop criterion separates self-bound packs from shares that can radiate: the horizon accepts only unpacked energy, and packing is completed at the inner interface $R_i = R_s/2$, with registration on the horizons.
5. Temperature matching requires the accretion and radiation of E_c capsule quanta to be last in, first out, with entropy changes $\Delta S_\pm = \pm 4\pi k_B(2N \pm 1)$, and the history returns with the radiation in the reverse order of packing. In a merger the common outer interface forms outside both; the smaller black hole dissolves while its E_c packs do not, and they are re-registered in their original order after those of the larger black hole, preserving the temperature ranking; the total number of relations before the merger, $(N_1 + N_2)^2$ (including the $2N_1N_2$ group relations), equals the $N_{irr,f}^2$ in the new black hole plus the relations of the $N_u < N_2$ packs leaving the irreducible part from the tail of the sequence, so information is conserved. A single E_c neither packs nor unpacks; the E_c in the cosmic background are stable, and there is no radiation from E_c in cosmic voids. The smallest black hole forms from two combining E_c, which requires a contact density reached only in the early universe; evaporation ends with the unpacking $2 \to 1$, which emits the last pack at the highest temperature of the whole process, 3.76×10^{30} K, leaving one E_c that records no history.
6. Compared with the Bekenstein–Mukhanov scheme, the area steps are likewise equally spaced, but $e^{4\pi}$ is not an integer and the area steps themselves give no lines of fixed energy. The end stage of evaporation is defined by k_BT exceeding the rest energy of the electron ($N \lesssim 9.5 \times 10^{20}$), after which the emitted particles include massive ones; the temperature staircase is significant only for $N \lesssim 10^2$, far beyond detectable energies; at the end each sub-step is shared by only a few particles, so the spectrum may approach discrete lines; the relational entropy itself has no logarithmic term.

The Page curve.

7. Conservation of relational entropy, $S_{BH} + S_{out} = S_0$, together with the unitarity of the overall evolution, gives the Page curve (summing the entropy released pack by pack gives the same result, without island or replica-wormhole constructions), with the information Page point $x_P = 1 - 1/\sqrt{2} \approx 0.293$, independent of β; the thermodynamic Page point defined with the coarse-grained radiation entropy comes earlier because of thermalization outside the horizon, and the difference reflects the entropy production of evaporation. The Page curve gives the amount of information; last-in-first-out gives its content.

The entropy of Kerr black holes.

8. For a Kerr black hole counted by its irreducible mass, the entropy decomposition holds unchanged with the capsule number N_{irr}, $S_{BH}/k_B = 4\pi N_{irr}^2$, and together with the energy formula of general relativity it reproduces the Kerr temperature; the sum of the two horizon areas is always $4\pi R_s^2$; the outer horizon carries the irreducible entropy $4\pi k_B N_{irr}^2$ and the inner horizon the rotational part $4\pi k_B(N^2 - N_{irr}^2)$, the total entropy being $4\pi k_B N^2$; in classical accretion, increasing the spin requires increasing the mass; absorbing an irreducible pack increases the entropies of both horizons (for $N_{rot} > 0$), absorbing a rotational pack increases only that of the inner horizon, the total entropy increase being the same for both routes, and the outer-horizon area never decreases; the share of the inner horizon increasing with spin; writing $N_{irr} = N\cos\omega$, $\sin 2\omega = \chi$, and the areal radii of the two horizons, $R_s\cos\omega$ and $R_s\sin\omega$, are the two legs of a right triangle with hypotenuse R_s; the 29.3% continuum-limit upper bound (not reached by black holes of finite N) on the extractable rotational energy corresponds to the balance point of the two horizon entropies, equal in value to the information Page point; the difference of the areal radii of the two horizons is at least $2\ell_p$, $2\ell_p$ being the lower limit, and the extremal Kerr state is the breaking point, consistent with the third law of black-hole thermodynamics; in the counting, $N = \sqrt{2}\,N_{irr}$ has no integer solution, and the allowed states are $\lceil(1 + \sqrt{2N^2 - 1})/2\rceil \le N_{irr} \le N$; black holes with $N \le 4$ do not rotate, and the smallest rotating state is $N = 5$, $N_{irr} = 4$.

These results use only the counting in N and thermodynamic identities. They are consistent with the corresponding established results—the Hawking and Kerr temperatures, the area and no-hair theorems, the Page curve and the Page point, Christodoulou's irreducible-mass relation and the third law of black-hole thermodynamics—and in addition give the counting origin of these results, together with content that Eq. (1) itself does not display, such as last-in-first-out, two-level quantization and the distribution of entropy over the two horizons.

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