The $2d_m$ Minimum Topological Structural Unit: From a Relational-Information Axiom to the Dark-Energy Capsule and the Cosmological Residual Potential

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Abstract

Quantum field theory with a Planck-scale cutoff overestimates the vacuum energy density by 120–123 orders of magnitude. Why does it not gravitate, and whence the small observed dark energy? Taking information as a measure of distinguishable relations between objects, we derive the $2d_m$ minimum topological structural unit axiom, corroborated by the Bekenstein–Hawking entropy and the holographic principle. Applied to the dark-energy capsule E_c and massless particles, it gives a common diameter $2\ell_p$ (capsule inner diameter) and energy ceiling E_P. Treating $\tau = c^4/(8\pi G)$ as a line energy density on spacetime interfaces, the total pull on the capsule's two interfaces is exactly the maximum force $c^4/(4G)$; it balances the outward push at radius ℓ_p, with the outer interface at the Schwarzschild radius $2\ell_p$ of E_P. For a Schwarzschild black hole of N capsule energies, the physical quantities, including the thermodynamic state functions, form a ladder from N^{-2} to N^3, and the entropy obeys $S_H/k_B = 4\pi N^2 = 4\pi[N + N(N-1)]$: self terms plus ordered pairwise relations, with no many-body remainder; so for black holes the entropy formula is an equality. After normalization over the capsule volume, the gravitational mass–energy limit and the quantum zero-point field agree exactly, defining an ideal symmetry point. The observed dark-energy density is the residual; its coefficient $\delta_c = \Omega_\Lambda/N_{\mathcal{H}}^2$ follows the same $1/N^2$ law ($N_{\mathcal{H}} \approx 4.25 \times 10^{60}$ for the Hubble sphere). Background capsules, whose positive zero-point energy and negative gravitational energy cancel, do not gravitate; their number in the Hubble sphere, $\sim 10^{181}$, exceeds its gravitating equivalent $N_{\mathcal{H}}$ by $\sim 10^{121}$, the vacuum-energy gap. A falsifiable creation-rate equation follows.

Keywords: Relational information, Minimum length, Black-hole entropy, Maximum force, Dark energy, Cosmological constant problem

1 Introduction

Quantum field theory estimates the vacuum zero-point energy with a cutoff at the Planck scale, and the resulting energy density exceeds the cosmologically observed value by about 120–123 orders of magnitude; this conflict is widely regarded as the most serious predictive failure of modern theoretical physics [1, 2]. At the same time, observations have firmly established that the expansion of the universe is accelerating. The Planck satellite finds that dark energy makes up about 68.5% of the total energy of the universe, and the dark-energy density derived from its parameters is $\rho_{\mathrm{obs}} = 5.25 \times 10^{-10}\,\mathrm{J\,m^{-3}}$ [3]. Baryon-acoustic-oscillation measurements by the Dark Energy Spectroscopic Instrument (DESI) have tested this result further and suggest that dark energy may evolve with time [4, 5]. The problem thus has two layers: why the enormous vacuum energy produces no corresponding gravitational effect (the "old" cosmological constant problem), and where the small observed dark energy comes from.

Several lines of research have addressed this paradox. Cohen, Kaplan and Nelson pointed out that the ultraviolet cutoff of an effective field theory should be limited by black-hole formation, i.e. by a holographic bound, which led to holographic dark-energy models [6, 7]. Causal-set theory starts from the fundamental discreteness of spacetime and regards the cosmological constant as a random fluctuation caused by discreteness: its mean is zero, its sign can reverse, and its magnitude always tracks the critical density, so that it naturally comes close to the observed value [8, 9]. The gravitational-aether proposal modifies the coupling between vacuum energy and gravity so that the vacuum energy density does not gravitate [10]. Wang, Zhu, Unruh and co-workers argue that the wildly fluctuating Planck-scale vacuum energy cancels in local oscillations and, through parametric resonance, leaves only a small net effect that drives the accelerated expansion [11–13]; the numerical conclusions were later revised by the same group, while the authors consider the basic mechanism still valid [12]. On another front, Jacobson, Verlinde and others seek the origin of gravity itself in horizon thermodynamics and entropy [14–17]. Each of these works has its strengths, but they leave a common gap. In many phenomenological models the Planck scale, as the minimum scale, enters as a cutoff or an assumption [18, 19] rather than being derived from a more basic principle. There is no unified answer to what structure exists at the minimum scale, or how such a structure could yield both the cutoff and the residual. Nor is there agreement on the residual itself: is it a random fluctuation, or a net difference with a definite direction? Why black-hole entropy scales with horizon area rather than volume is usually accepted as the starting point of the holographic principle, and the relational structure behind it is seldom discussed. Relatedly, the maximum force $c^4/(4G)$ is mostly put forward in the literature as a conjecture or principle; it is generally thought to be closely tied to horizons, and its universality is still debated [20–22].

This work tries to close this gap from a more basic starting point: information is not an intrinsic property of an isolated object but a measure of the distinguishable relations between objects. Consequently, the minimum information unit that any discrete system can hold must correspond to the minimum span required for two indivisible minimal units to establish a relation for the first time, namely $2d_m$. This is a common-sense corollary independent of any specific physical theory. It does not presuppose a minimum length; it derives a minimum distinguishable span. Nor does it depend on what physical quantity d_m represents: it may be a spatial scale, an energy, a momentum, or the dark-energy capsule unit discussed in this paper. Among existing approaches, this work is closest to the discreteness idea of causal sets [8, 9]: it likewise starts from the smallest discrete unit of spacetime, but instead of estimating the magnitude of the residual coefficient from statistical fluctuations, it uses the $2d_m$ axiom to fix the volume of the minimal capsule and performs a full topological normalization of the vacuum energy density. On this basis we re-express the Schwarzschild black hole with E_c as the counting unit, and find that the black-hole entropy decomposes exactly into the sum of the self terms of the individual capsules and the terms of their pairwise (ordered) relations, with no many-body remainder, and that for black holes the entropy formula can only hold as an equality: every capsule and every relation between a pair of capsules is established and fully counted, whereas the inequality of entropy bounds applies only to other systems, in which not all relations are established. This explains the area scaling of the entropy and provides a direct correspondence, from black holes, for the statement that "information is relation".

Section 2 derives the $2d_m$ axiom and uses the Bekenstein–Hawking entropy and the holographic principle as a cross-scale theoretical corroboration. Section 3 applies the axiom to the dark-energy capsule unit and to massless particles, obtaining the common equivalent diameter $2\ell_p$ and the common energy ceiling E_P; it interprets the inverse coupling constant of the field equations, $\tau = c^4/(8\pi G)$, as a line energy density on spacetime interfaces, and derives the maximum force and the equilibrium radius ℓ_p from the balance between the spacetime interfacial pull and the capsule's push, without assuming the maximum-force conjecture (Section 3.5). Section 4 describes the Schwarzschild black hole with E_c as the counting unit, gives the N-scaling ladder, and proves that the black-hole entropy decomposes exactly into capsule self terms and pairwise permutation (ordered-relation) terms, in strict correspondence with the $2d_m$ axiom; the cosmological residual coefficient obeys the same $1/N^2$ law; and the ratio $\phi N_{\mathcal{H}}^2 \sim 10^{121}$ of the background-capsule count in the Hubble sphere to its gravitating mass–energy equivalent gives the origin of the vacuum-energy gap. Section 5 uses a full topological normalization over the capsule volume to bring the gravitational mass–energy limit and the quantum zero-point field into exact 1:1 equality, and interprets the observed dark-energy density (ρ_{obs}) as a phase-transition residual potential away from the ideal symmetry point; background capsules do not gravitate, because their positive zero-point energy and negative gravitational energy cancel in a zero sum. Section 6 outlines, as a working hypothesis, the continuous creation of background capsules (creation proceeds on the basis of a constant capsule density ρ_c; the constancy of the residual coefficient δ_c is a setting consistent with current observations (ΛCDM)), and gives a falsifiable creation-rate equation tied to the Hubble parameter.

2 The $2d_m$ minimum topological structural unit axiom

2.1 The relational nature of information

Information is a distinguishable state produced by relations; its capacity is determined by the relational complexity within a system, not by the number of objects (this view was inspired by Shannon [23]). This view, that capacity is set by the internal relational complexity of a system, is akin in spirit to the treatment in statistical mechanics of entropy as a statistical property of the system as a whole rather than a property of a single particle [24]. The proposition can be established from the most basic common sense.

A completely isolated object, although it can exist independently, has no relational properties at all: it cannot produce with itself any distinguishable relation such as left and right, front and back, above and below, adjacency, distance, direction or coincidence. An isolated object therefore cannot carry any information:

$$\text{Capacity(isolated object)} = 0 \text{ bit} \tag{1}$$

Relations can arise only from the coexistence of two or more objects. Information is therefore not, at bottom, an intrinsic property of an isolated object, but a property of the system that emerges only after relations have formed.

2.2 The existence threshold and the $2d_m$ axiom

Let d be a physical quantity that is linearly additive and has an intrinsic minimum d_m: no independently existing value of d is smaller than d_m. Since d_m is defined as the indivisible minimal unit, two independent d_m units cannot partially overlap. The possible spans between two d_m units can therefore be divided strictly into three regimes:

- span $< 2d_m$: an unrealizable configuration (the relation has not yet come into being);
- span $= 2d_m$: the unique, minimal realizable configuration (the relation comes into being for the first time);
- span $> 2d_m$ (e.g. $2.1d_m$, $2.5d_m$, $4.6d_m$, $8d_m$, ...): the relation already exists, and the span becomes a free, continuous variable.

In other words, $2d_m$ is not the "first step" of some continuous, countable sequence of lattice points, but an insurmountable existence boundary: below it no structure is possible, above it lies the continuum. We thereby establish the equivalence between the minimum topological structural unit and the minimum information capacity, which is the core axiom of this paper:

$$\begin{gathered}\textbf{Minimum topological structural unit } (2d_m) \\ \Updownarrow \\ \textbf{1 bit of relational freedom}\end{gathered} \tag{2}$$

2.3 Universality of d

The derivation above depends only on d having an intrinsic minimum d_m and being linearly additive, not on what physical quantity d represents. Hence d may stand for any such quantity: $d = E$ for energy, $d = P$ for momentum, and $d = C$ for the dark-energy capsule unit discussed in Section 3. This universality means that the $2d_m$ axiom applies not only to space itself but also provides a unified theoretical basis for its later application to the capsule structure and to massless particles.

In principle, all generalizations in this section are restricted to physical quantities that are strictly linearly additive, i.e. for which $q(A \cup B) = q(A) + q(B)$ holds exactly (such as length, area or volume in flat space, or energy in the weak-interaction limit). This strict additivity can also be relaxed naturally in the sense of a local limit: for any well-behaved (smooth) physical quantity, its variation within a sufficiently small neighbourhood is always approximately linear at the leading order of its Taylor expansion, a default result guaranteed by the condition that the local range tends to its minimum. All quantitative applications of the $2d_m$ axiom in this paper (the axiomatic derivation and capsule construction in Sections 2 and 3, and the density calculations derived from them in Section 5) are strictly confined to the physically attainable minimum limiting range from ℓ_p to $2\ell_p$; in this sense the linear approximation is satisfied and need not be justified separately for each physical quantity.

When d is a length, we take d_m to be the Planck length (ℓ_p), so that $2d_m = 2\ell_p$. Taking the Planck length as the minimum length is consistent with studies of a minimal length in quantum gravity [18, 19]. This "existence-threshold" framework has the same logical structure as Bekenstein's classic thought experiment of dropping 1 bit of information into a black hole. Specializing the axiom to the spatial scale gives

$$\begin{array}{c} \text{Minimum topological structural unit } (2d_m = 2\ell_p) \\ \Updownarrow \\ \text{1 bit of relational freedom} \end{array} \tag{3}$$

Applying this existence threshold simultaneously along two independent orthogonal directions (i.e. taking the Cartesian product) extends the one-dimensional case naturally to two dimensions:

$$2d_m \times 2d_m = 2\ell_p \times 2\ell_p = 4\ell_p^2 \tag{4}$$

This logical structure and the two-dimensional area quantum together form the basis of the theoretical corroboration in Section 2.4.

2.4 Black-hole thermodynamics as an independent theoretical corroboration

Sections 2.1–2.3 derived the $2d_m$ axiom independently from the general principle of relational information. This section shows further that the derivation is fully consistent with one of the most rigorous results of information physics in quantum gravity, the

Bekenstein–Hawking black-hole entropy formula, which thus provides the $2d_m$ axiom with an independent theoretical corroboration from extreme physical conditions.

When the $2d_m$ axiom is applied to the extreme physical boundary of a black-hole horizon ($d_m = \ell_p$), the existence criterion "has the relation come into being?" corresponds exactly to "has the horizon area jumped by $4\ell_p^2$?". In the Bekenstein–Hawking entropy formula [25, 26],

$$S = \frac{k_B A}{4\ell_p^2} \tag{5}$$

where A is the horizon area of the black hole, ℓ_p is the Planck length and ℓ_p^2 is one Planck area. Bekenstein proposed that black-hole entropy is proportional to the horizon area [25], and Hawking fixed the coefficient 1/4 through black-hole radiation [26]; for black holes, Eq. (5) is an equality. A separate class of entropy bounds (the holographic bound) gives, for general physical systems, the inequality $S \leq k_B A/(4\ell_p^2)$, with A the area enclosing the system [27]; the black hole is the system for which the bound becomes an equality. Section 4.3 will show that, with E_c as the counting unit, the equality is the only option for black holes, and the inequality applies only to other systems. Entropy is measured with the natural logarithm ($S = k_B \ln W$), so Eq. (5) means that every four Planck areas correspond to one natural unit of entropy (k_B, i.e. 1 nat $\approx$ 1.44 bits). Expressed in the information-theoretic unit, the bit, storing 1 bit means an entropy increase of $k_B \ln 2$. When Bekenstein drops "the smallest unit of information (1 bit)" into a black hole, Eq. (5) gives the minimum increase of the horizon area as

$$\Delta A = 4\ln(2)\,\ell_p^2 \approx 2.77\,\ell_p^2 \tag{6}$$

Note that the coefficient 1/4 in Eq. (5) is the exact value given by Hawking; Bekenstein's original thought experiment gave only an order-of-magnitude estimate. We take the bit as the starting point because the $2d_m$ axiom describes the binary distinction "relation present or absent", whose information content is naturally one bit; the natural unit (nat) is measured with the natural logarithm and is not a counting unit for a discrete structure. The two connect as follows. By the $2d_m$ axiom the minimum distinguishable span is $2\ell_p$, so the smallest admissible area cell is the $4\ell_p^2$ of Eq. (4); no smaller area cell exists in this framework. By Eq. (5) this cell has an entropy capacity of 1 nat $\approx$ 1.44 bits: enough to store 1 bit (which requires $4\ln(2)\,\ell_p^2 \approx 2.77\,\ell_p^2$) but not 2 bits (which require $8\ln(2)\,\ell_p^2 \approx 5.55\,\ell_p^2$). Hence, within the $2d_m$ framework, $4\ell_p^2$ is precisely the smallest area cell that can hold one complete bit, and it is in this sense that "storing 1 bit requires at least 4 Planck areas" holds.

It should be stressed that the validity of the $2d_m$ axiom does not depend on black-hole physics or on any theory of quantum gravity; it is a logical consequence of the relational nature of information in any discrete space. Black-hole thermodynamics plays here only the role of a cross-scale theoretical corroboration. It shows that even at the physical boundary where information density is highest and spacetime curvature most extreme, the common-sense principle that "information is relation" still holds strictly, and gives the same minimum scale ($4\ell_p^2 = 2\ell_p \times 2\ell_p$) as the holographic principle [27–29], strongly supporting the universality of the $2d_m$ axiom across scales.

Section 4.3 will show further that, with E_c as the counting unit, the black-hole entropy as a whole decomposes exactly into the self terms of the individual units and their pairwise relational terms, with no many-body remainder, which shows that for black holes Eq. (5) can only hold with equality.

3 The dark-energy capsule unit E_c

3.1 Inner diameter of the capsule: a direct application of the $2d_m$ axiom

Following Section 2.3, we apply the $2d_m$ axiom to the physical object $d = C$ (the dark-energy capsule unit). As an indivisible minimal topological structure, the capsule must be able to accommodate, in its interior, a distinguishable relation established for the first time, so its minimum span must likewise be $2d_m$. The inner diameter of the capsule, D_i, is therefore given directly by the axiom:

$$D_i = 2d_m = 2\ell_p \tag{7}$$

Taking the wall thickness δR at its smallest possible value, $\delta R = \ell_p$, the outer diameter D_o follows from the inner diameter plus the wall thickness on both sides:

$$D_o = D_i + 2\delta R = 2\ell_p + 2\ell_p = 4\ell_p \tag{8}$$

Correspondingly, the outer radius is $R_o = D_o/2 = 2\ell_p$, and the topological volume of the capsule is taken to be spherical:

$$V_c = \frac{4}{3}\pi R_o^3 = \frac{4}{3}\pi(2\ell_p)^3 = \frac{32\pi}{3}\ell_p^3 \tag{9}$$

3.2 Generalized diameter and wavelength of massless particles

Applying the $2d_m$ axiom in the same way to any massless particle: as an indivisible entity, the minimum diameter required for its existence, D_γ, is likewise given directly by the axiom:

$$D_\gamma = 2d_m = 2\ell_p \tag{10}$$

The corresponding minimum wavelength then follows geometrically. For a circle (or sphere) of diameter $2\ell_p$, the path traced by any point on the circumference in one complete rotation about the centre has length $\pi\times(2\ell_p) = 2\pi\ell_p$; this is also the distance the centre of a sphere of diameter $2\ell_p$ advances when it rolls one full turn without slipping (path length of one turn = $\pi\times$ diameter = $2\pi\ell_p$). Thus

$$\lambda_{\min} = 2\pi\ell_p \tag{11}$$

Equation (11) is the standing-wave condition on the minimal closed path: the circumference $2\pi\ell_p$ of the minimal structure of diameter $2\ell_p$ is the minimal closed path, and the ground-state wavelength closes on it exactly once. This is Bohr's quantization condition $n\lambda = 2\pi r$ (in de Broglie form) with $n = 1$ and $r = \ell_p$ [30]; the sphere rolling one

turn without slipping is its intuitive picture. The derivation involves no specific detail of the electromagnetic interaction. The diameter and wavelength obtained above are therefore not specific to photons and apply in principle to any massless boson (such as gluons and the hypothetical graviton). The photon is simply one concrete, observable example, and we use it as the example in what follows.

It should be noted that Eqs. (10) and (11) hold simultaneously only at the $2d_m$ limit point. For a massless particle with energy below E_P, the core diameter remains $2d_m = 2\ell_p$ and never changes; only the wavelength varies with energy ($\lambda = 2\pi\hbar c/E$).

3.3 Scale coincidence of the E_c capsule and massless particles

Comparing the results of Sections 3.1 and 3.2, the inner diameter of the capsule and the equivalent diameter of a massless particle are both $2d_m = 2\ell_p$. This is not the result of designing one to fit the other; it is a natural coincidence obtained when the same $2d_m$ axiom acts independently on two different physical objects (one a hollow topological container, the other an indivisible entity). The coincidence occurs at the level of the axiom, before any specific energy calculation.

3.4 Independent cross-check of the energy scale

Zero-point energy of the capsule unit. Extending the $2d_m$ framework to the energy parameter ($d = E$), the zero-point energy per unit is $E_m = (1/2)\hbar\omega_p$, and the total zero-point energy of the capsule is $E_0 = 2E_m = \hbar\omega_p$. Using the relation between the Planck angular frequency and the Planck length, $\omega_p = c/\ell_p$, we obtain

$$E_0 = 2E_m = \hbar\omega_p = \frac{\hbar c}{\ell_p} \equiv E_P \tag{12}$$

Here $E_P \equiv m_P c^2 = \hbar c/\ell_p$ is exactly the standard Planck energy: the zero-point energy of the capsule unit equals the standard Planck energy. The factor 2 comes from the $2d_m$ axiom: the capsule span $2d_m$ contains two minimal units d_m, each with zero-point energy $E_m = (1/2)\hbar\omega_p$, giving $E_0 = 2E_m$.

Energy of the highest-energy photon. A photon of wavelength $\lambda_{\min} = 2\pi\ell_p$ has energy

$$E_{\gamma,\max} = \frac{hc}{\lambda_{\min}} = \frac{2\pi\hbar c}{2\pi\ell_p} = \frac{\hbar c}{\ell_p} = E_P \tag{13}$$

where $h = 2\pi\hbar$. Thus two calculations for different physical objects, (a) the zero-point energy of the capsule unit, $E_0 = 2E_m$, and (b) the energy of the highest-energy photon, $E_{\gamma,\max}$, obtained from the geometric argument of Section 3.2, each give the same value E_P on their own, rather than two numbers being tuned to agree:

$$E_0 = 2E_m = E_{\gamma,\max} = E_P \tag{14}$$

This result carries a striking contrast: several "limits" are locked together here at once. The smallest, indivisible topological structural unit E_c, its minimum volume V_c, and the lowest (zero-point) energy it carries, $E_0 = 2E_m$ — these three "minimum"

quantities together determine exactly the highest-energy, shortest-wavelength photon energy $E_{\gamma,\max}$ of the whole framework. Three lower bounds "hold up" a single upper bound, and all four converge on the same value E_P, neither by coincidence nor by parameter tuning. This further underlines the special status of the Planck scale as the unique natural energy scale of the universe.

3.5 Boundary constraint of the E_c capsule and the maximum force

The lower bound on the capsule size is given by the $2d_m$ axiom and does not rely on any force-balance calculation. Below we use force balance to test whether the structure fixed by the axiom is mechanically self-consistent, in six steps.

(1) From the field-equation coupling constant to spacetime interfacial tension. Einstein's field equations read $G_{\mu\nu} = (8\pi G/c^4)\, T_{\mu\nu}$. The Einstein tensor $G_{\mu\nu}$ on the left describes the curvature of spacetime and has the unit of curvature, m^{-2}; the energy–momentum tensor $T_{\mu\nu}$ on the right describes the distribution of energy and momentum and has the unit of energy density, $\mathrm{J\,m}^{-3}$; the coefficient $8\pi G/c^4$ links the two. Multiplying both sides by the reciprocal of this coefficient, the field equations can be written as

$$T_{\mu\nu} = \tau\, G_{\mu\nu}, \qquad \tau \equiv \frac{c^4}{8\pi G} \tag{15}$$

The unit of τ follows from the ratio of the units of the two sides of Eq. (15): $(\mathrm{J\,m}^{-3})/(\mathrm{m}^{-2}) = \mathrm{J\,m}^{-1}$, i.e. energy per unit length; its value is about $\tau = 4.8 \times 10^{42}\,\mathrm{J\,m}^{-1}$. Regarding c^4/G as the "stiffness" with which spacetime resists bending has a precedent in induced gravity [31]. Einstein's field equations can also be derived from horizon thermodynamics or from entanglement equilibrium [14, 15], which shows that the field equations themselves can be understood from the viewpoint of horizons. In the present model τ is interpreted as a line energy density on a spacetime interface. To avoid confusion with the wavelength λ of Section 3.2, we denote this line energy density by τ, a notation consistent with the string tension of the cosmic-string literature.

We now convert the line energy density τ into a surface energy density, i.e. a spacetime interfacial tension (to be distinguished from the interfacial tension of ordinary materials). Consider a spherical interface of radius R, along one of whose great circles (circumference $2\pi R$) lies a string of line energy density τ; the energy of this string is $2\pi R\tau$. Distributing this energy uniformly over the whole sphere (area $4\pi R^2$) gives the energy per unit area, i.e. the spacetime interfacial tension:

$$\sigma(R) = \frac{2\pi R\tau}{4\pi R^2} = \frac{\tau}{2R} \tag{16}$$

Its unit is $\mathrm{J\,m}^{-2} = \mathrm{N\,m}^{-1}$, the same as that of ordinary surface tension. σ is the energy required per unit increase of interface area, $\mathrm{d}E/\mathrm{d}A$. As a side check, the energy of a Schwarzschild horizon is $E = Rc^4/(2G)$ and its area is $A = 4\pi R^2$, so $\mathrm{d}E/\mathrm{d}A = (\mathrm{d}E/\mathrm{d}R)/(\mathrm{d}A/\mathrm{d}R) = [c^4/(2G)]/(8\pi R) = c^4/(16\pi GR)$, while Eq. (16) gives $\sigma(R) = \tau/(2R) = c^4/(16\pi GR)$; the two are identical, i.e. Eq. (16) equals the energy required per unit increase of horizon area in the first law of black-hole mechanics [32, 33].

(2) Pull force of a spherical cavity. Let a spherical cavity have radius R, and cut it along its equatorial plane into two hemispheres, which are then connected only through the great circle of the cut. The tension per unit length along the cut is $\sigma(R)$ and the length of the cut is $2\pi R$, so the force pulling the two hemispheres together is

$$F_{\text{pull}}(R) = 2\pi R \times \sigma(R) = 2\pi R \times \frac{\tau}{2R} = \pi\tau \tag{17}$$

The radius R cancels completely in Eq. (17): the pull force does not depend on the size of the interface.

Equations (15)–(17) use only τ and spherical geometry and involve no specific property of black holes or of the capsule; in the present model they apply, as a model assumption, to a spacetime interface of any radius R. Note that the spacetime interfacial tension originates in spacetime geometry itself, whereas the tension of an ordinary material interface (such as a liquid surface) originates in intermolecular interactions; the two have different origins and must not be conflated.

(3) Pull force of the inner and outer interfaces of the E_c capsule. Cut the capsule along one equatorial plane; the cut passes through both the inner and the outer interface circles. By Eq. (17) the pull force is independent of radius, so the two interfaces exert the same pull force:

$$F_{\text{pull}}(R_o) = F_{\text{pull}}(R_i) = \pi\tau \tag{18}$$

The spacetime interfacial tensions on the two interface circles act in the same direction, both pulling the two hemispheres together. By the $2d_m$ axiom the spacing between the inner and outer interfaces cannot be smaller than ℓ_p and cannot be compressed further, so the inward pull of the outer interface is transmitted through the intervening layer to the inner interface. The two pull forces act on the same body, and the total pull force is their sum:

$$F_{\text{pull,total}} = F_{\text{pull}}(R_o) + F_{\text{pull}}(R_i) = \pi\tau + \pi\tau = 2\pi\tau \tag{19}$$

Substituting the expression for τ from Eq. (15):

$$F_{\text{pull,total}} = 2\pi \times \frac{c^4}{8\pi G} = \frac{c^4}{4G} = F_{\max} \tag{20}$$

This value is exactly the maximum-force conjecture value $F_{\max} = c^4/(4G)$ in the literature [20, 21, 34, 35]. Three points should be stressed. First, this work does not take the maximum-force conjecture as a premise: the total pull force is derived from the field-equation coupling constant and spherical geometry, and the agreement is a result of the derivation. Second, since R cancels completely in Eq. (17), the total pull force is a scale-independent constant. Third, regarding the debate on universality mentioned in the Introduction, the counterexamples given by Jowsey and Visser [36] concern only idealized static fluid spheres close to the collapse threshold (the Buchdahl limit), not horizons themselves, and therefore do not affect the conclusions here.

In the literature this value has been obtained along two main routes. First, Gibbons obtained it in specific general-relativistic settings: for example, the tension of a strut holding two black holes apart [20] and the tension of a cosmic string [20, 34] cannot exceed $c^4/(4G)$. Second, Schiller linked it to horizons, which are precisely the systems that realize this bound: the maximum force equals the energy of a Schwarzschild black hole divided by its diameter [21, 35]. Restated in terms of the surface gravity, i.e. the gravitational acceleration at the horizon: at the Schwarzschild radius $R_s = 2GM/c^2$ it is $\kappa = GM/R_s^2 = c^4/(4GM)$, and multiplying by the mass gives $M\kappa = c^4/(4G)$; equivalently, an energy divided by its Schwarzschild diameter gives $E/D_s = c^4/(4G)$ (see Eq. (27)). The latter route requires a convention on *where* the acceleration multiplying the mass is evaluated: at the centre of mass the gravitational field of a symmetric body vanishes and the result is zero; at the horizon, the mass is not located there and κ is a quantity redshifted to infinity, so the route is better regarded as an order-of-magnitude argument. The route taken here differs from both. It uses only the field-equation coupling constant τ and spherical geometry, and computes the resultant of the spacetime interfacial tension across an equatorial cut. The force is defined unambiguously (across the cut, acting on the interfaces), and neither a mass and its location nor an evaluation point for an acceleration needs to be introduced; the only ingredients are the field equations, the conversion between line and surface energy density, and the force across a cut, all of which are elementary physical notions. That the two kinds of route give the same value at a horizon indicates that the spacetime interfacial pull is precisely the geometric expression of gravitational confinement at the horizon.

(4) Outward push of the energy inside the E_c capsule. We first fix the relation between pressure and energy density. The capsule energy comes from the zero-point field ($\hbar\omega_p$) and equals the energy of the highest-energy massless particle (Section 3.4); it is the energy of a massless field. Massless energy moves at the speed of light in all directions; for an isotropic distribution the energy density is shared equally among three mutually perpendicular directions, so the pressure in each direction is one third of the energy density, $p = \rho/3$. This is equivalent to the vanishing of the trace of the energy–momentum tensor $T_{\mu\nu}$ in Eq. (15): in the rest frame of the medium, $T^{\mu}_{\mu} = -\rho + 3p = 0$ [37]. It depends only on the condition of masslessness and does not require the capsule literally to contain a photon.

Let the capsule energy E_0 be distributed uniformly within a sphere of radius R, where R is the radius to be determined. The energy density $\rho(R)$ and the pressure $p_{\text{push}}(R)$ are

$$\rho(R) = \frac{E_0}{V} = \frac{E_0}{(4/3)\pi R^3} \tag{21}$$

$$p_{\text{push}}(R) = \frac{\rho}{3} = \frac{E_0}{3 \times (4/3)\pi R^3} = \frac{E_0}{4\pi R^3} \tag{22}$$

If the sphere is cut into two halves along the equatorial plane, the pressure acts on the equatorial cross-section (area πR^2), and the force pushing the two halves apart is

$$F_{\text{push}}(R) = p_{\text{push}} \times \pi R^2 = \frac{E_0}{4\pi R^3} \times \pi R^2 = \frac{E_0}{4R} \tag{23}$$

(5) Force balance: solving for the equilibrium radius. The push force decreases as the radius grows, whereas the total pull force is a radius-independent constant. Only the spacetime interfacial pull is taken as the inward force here; the self-gravity of the capsule is not added separately. The capsule energy is zero-point energy, and in the present framework (Section 5.3) its gravitational effect couples to spacetime geometry through the interfaces and appears as the spacetime interfacial pull, not as an additional Newtonian self-gravity in the bulk; moreover, inside the capsule $R < R_s$, where the weak-field Newtonian approximation no longer applies anyway. Setting the push force (Eq. (23)) equal to the total pull force (Eq. (19)) and denoting the equilibrium radius by $R = R_{\rm eq}$:

$$\frac{E_0}{4R_{\rm eq}} = 2\pi\tau \;\Rightarrow\; R_{\rm eq} = \frac{E_0}{8\pi\tau} = E_0 \times \frac{8\pi G}{8\pi c^4} = \frac{GE_0}{c^4} \tag{24}$$

Using the relations among Planck quantities, $E_0 = E_P = \sqrt{\hbar c^5/G}$ and $\ell_p = \sqrt{\hbar G/c^3}$, their ratio is $E_P/\ell_p = \sqrt{(\hbar c^5/G) \times (c^3/(\hbar G))} = c^4/G$, i.e. $E_P = (c^4/G)\,\ell_p$. Substituting into Eq. (24):

$$R_{\rm eq} = \frac{GE_0}{c^4} = \frac{GE_P}{c^4} = \ell_p \tag{25}$$

The equilibrium radius obtained from force balance, $R_{\rm eq} = \ell_p$, coincides exactly with the inner-interface radius $R_i = D_i/2 = \ell_p$ given independently by the $2d_m$ axiom in Section 3.1. Adding the layer thickness $\delta R = \ell_p$ given by the axiom, the outer interface lies at $R_o = R_i + \delta R = 2\ell_p = 2GE_P/c^4 = R_s$, exactly the Schwarzschild radius (R_s) of the Planck energy. Two mutually independent routes (the geometric construction of the axiom and the force balance of spacetime interfacial tension) give the same geometric structure, and the outer interface coincides with the Schwarzschild horizon. At the equilibrium radius the push force of the capsule is

$$F_{\rm push}(R_{\rm eq}) = \frac{E_P}{4\ell_p} = \frac{1}{4} \times \frac{E_P}{\ell_p} = \frac{c^4}{4G} = F_{\rm max} \tag{26}$$

That is, the push produced by the energy inside the capsule is exactly the maximum force, and it is exactly balanced by the total pull force $2\pi\tau = F_{\rm max}$ of the two interfaces: a capsule of Planck energy requires precisely the maximum force to confine it.

This equilibrium is stable. The push force $F_{\rm push}(R) = E_0/(4R)$ decreases with increasing radius, while the total pull force $2\pi\tau$ is independent of radius. The equilibrium radius lies exactly at the existence lower bound given by the $2d_m$ axiom (inner radius not smaller than ℓ_p): inward compression is forbidden by the axiom; if the interface is displaced outward ($R > R_{\rm eq}$), the push is smaller than the total pull and the interface is pulled back to the equilibrium point. Hence, for a fixed capsule energy E_0, the capsule is in stable equilibrium.

(6) Conversion between energy and force. Mass and energy are converted by c^2; likewise, energy and force can be converted by a length. From Eqs. (20) and (26),

$F_{\max} = E_P/(4\ell_p)$, and $4\ell_p$ is precisely the outer diameter of the capsule, D_o; hence

$$F_{\max} \times D_o = \frac{c^4}{4G} \times 4\ell_p = \frac{c^4}{G}\,\ell_p = E_P \tag{27}$$

i.e. the maximum force equals the capsule energy divided by the capsule's outer diameter.

In general, the Schwarzschild diameter of an arbitrary energy E is $D_s = 4GE/c^4$, and $E/D_s = c^4/(4G)$, independent of E. The maximum force can therefore be understood as an upper bound on the concentration of energy: a quantity of energy cannot be released entirely over a distance shorter than its own Schwarzschild diameter.

Taking $2d_m = 2\ell_p$ as the minimum relational string length, the minimum string energy is $\tau \times 2\ell_p = [c^4/(8\pi G)] \times 2\ell_p = (c^4/G)\,\ell_p/(4\pi) = E_P/(4\pi) = E_0/(4\pi)$, so the capsule energy E_0 is exactly 4π times this minimum string energy. This relation can be understood as follows: the minimum string energy corresponds to the energy per steradian, and covering all 4π steradians gives the full capsule energy E_0; the factor 4π comes from the spherical geometric factor contained in the coupling constant $8\pi G$ of the field equations.

Two remarks are in order. First, the precise condition for $p = \rho/3$ is that the particle energy far exceeds its rest energy ($E \gg mc^2$): the rest mass can then be neglected, the momentum is approximately E/c, and any particle behaves like a massless one. At the Planck scale this condition is satisfied, so $p = \rho/3$ inside the capsule does not depend on the species of particles in it. The condition may also hold near the inner horizon of macroscopic black holes: radiation falling toward the inner horizon is continually blueshifted, and the energy in that region is dominated by ultra-relativistic radiation [38, 39], consistent with $p \approx \rho/3$. Note, however, that if E_0 in Eqs. (21)–(24) is replaced by an arbitrary energy E, the steps of the derivation are unchanged and the equilibrium radius is GE/c^4, half the Schwarzschild radius; in general it does not coincide with the inner horizon of the Kerr solution. Whether the present force balance extends to macroscopic black holes is left for future work. Second, $p = \rho/3$ describes the internal stress of a single bounded capsule, which is completely enclosed by the spacetime interfacial tension and does not appear as a macroscopic bulk pressure; the dark-energy equation of state $w \approx -1$ in Section 6 describes the macroscopic behaviour of the capsule ensemble, whose background mean density is kept constant by continuous creation during expansion. How the two connect is likewise left for future work.

4 From E_c to black holes: N-scaling and the relational decomposition of black-hole entropy

Section 3 fixed the structure of a single E_c capsule. This section considers the case in which the energies of N capsules are concentrated in one place, i.e. it describes a Schwarzschild black hole with E_c as the counting unit. Only the Schwarzschild radius and the Bekenstein–Hawking entropy formula are used; the interior structure of the black hole is not involved, and the rotating (Kerr) case is left for future work.

4.1 Black holes in units of E_c: $D_H = N \cdot D_o$

Section 3.5 showed that the radius of the outer interface of the E_c capsule, $R_o = 2\ell_p$, is exactly the Schwarzschild radius of the Planck energy. Let the total energy of a Schwarzschild black hole be the sum of N capsule energies, $E = NE_0 = NE_P$. Its Schwarzschild radius is

$$R_s(N) = \frac{2GE}{c^4} = \frac{2GNE_P}{c^4} = N \cdot \frac{2GE_P}{c^4} = N \cdot R_o \tag{28}$$

where $R_o = 2GE_P/c^4 = 2\ell_p$. Correspondingly, the horizon diameter is

$$D_H = 2R_s(N) = N \cdot (2R_o) = N \cdot D_o \tag{29}$$

That is, the horizon diameter of a black hole containing N capsule energies is exactly the length of N capsule outer diameters $D_o = 4\ell_p$ (Eq. (8)) laid end to end. It should be stressed that Eq. (29) follows only from the proportionality of the Schwarzschild radius to the energy; it does not imply that the capsules are arranged in a line inside the horizon, and it says nothing about how the capsules are distributed in the black-hole interior.

For $N = 1$, $D_H = D_o$: the outer interface of E_c coincides exactly with the Schwarzschild radius of the Planck energy, which is the limit point of the sequence. By the $2d_m$ axiom, a relation requires at least two indivisible minimal units (Section 2.2); we therefore take $N = 2$ as the starting point of the black-hole sequence: the smallest black hole consists of the energies of two capsules, and E_c itself is not a black hole. Section 4.3 gives a second reason for this conclusion from the decomposition of the entropy.

4.2 The N-scaling ladder

By Eq. (29), several physical quantities of the black hole can be written as the corresponding quantity of a single E_c capsule multiplied by an integer power of N. We derive them one by one.

(1) **Horizon area A_H and horizon volume V_H.** The horizon area is $A_H = \pi D_H^2 = N^2 \cdot \pi D_o^2$, where $A_o = \pi D_o^2 = 16\pi\ell_p^2$ is the area of the capsule's outer interface. The volume enclosed by the horizon (the Euclidean equivalent volume defined by the areal radius) is $V_H = (4/3)\pi R_H^3 = N^3 \cdot (4/3)\pi R_o^3$, and $(4/3)\pi R_o^3$ is exactly the capsule volume V_c of Eq. (9). Hence

$$A_H = N^2 A_o, \qquad V_H = N^3 V_c \tag{30}$$

(2) **Mean energy density ρ_H.** $\rho_H = E/V_H = NE_P/(N^3 V_c) = (E_P/V_c)/N^2$. Writing the capsule energy density as $\rho_c \equiv E_P/V_c$, we have

$$\rho_H = \frac{\rho_c}{N^2} \tag{31}$$

(3) **Hawking temperature T_H.** The Hawking temperature of a Schwarzschild black hole is $k_B T_H = \hbar c^3/(8\pi GM)$. Substituting $M = Nm_P$ and using $\hbar c^3/(Gm_P) = (\hbar c^3/G)/\sqrt{\hbar c/G} = \sqrt{\hbar c/G}\cdot c^2 = m_P c^2 = E_P$, we obtain

$$k_B T_H = \frac{E_P}{8\pi N} \tag{32}$$

(4) **Maximum force $F_{\max}$.** The ratio of energy to horizon diameter is $E/D_H = NE_P/(ND_o) = E_P/D_o = c^4/(4G) = F_{\max}$, independent of N; this is Eq. (27).

(5) **Entropy S_H.** By Eq. (5), $S_H/k_B = A_H/(4\ell_p^2) = N^2 A_o/(4\ell_p^2) = 4\pi N^2(2\ell_p)^2/(4\ell_p^2)$, i.e.

$$\frac{S_H}{k_B} = 4\pi N^2 \tag{33}$$

(6) **Heat capacity C.** By Eq. (32), $N = E_P/(8\pi k_B T_H)$, so the internal energy is $U = NE_P = E_P^2/(8\pi k_B T_H)$. Differentiating with respect to T_H gives $\mathrm{d}U/\mathrm{d}T_H = -E_P^2/(8\pi k_B T_H^2)$; by Eq. (32) again, $k_B^2 T_H^2 = E_P^2/(64\pi^2 N^2)$, and substitution gives

$$C = \frac{\mathrm{d}U}{\mathrm{d}T_H} = -\frac{E_P^2 k_B}{8\pi k_B^2 T_H^2} = -\frac{E_P^2 k_B \cdot 64\pi^2 N^2}{8\pi E_P^2} = -8\pi k_B N^2 = -2S_H \tag{34}$$

The heat capacity is negative, scales as N^2, and equals -2 times the entropy.

(7) **Internal energy, free energy and enthalpy.** The internal energy is the total energy of the black hole, $U = E = NE_P$. By Eqs. (32) and (33),

$$T_H S_H = \frac{E_P}{8\pi N}\cdot 4\pi N^2 = \frac{NE_P}{2} = \frac{U}{2} \tag{35}$$

which is the Smarr relation $U = 2T_H S_H$ [33]. The Helmholtz free energy is

$$F = U - T_H S_H = NE_P - \frac{NE_P}{2} = \frac{NE_P}{2} \tag{36}$$

A Schwarzschild black hole lives in asymptotically flat spacetime with no external pressure, so the pV term of the enthalpy vanishes and the enthalpy equals $U = NE_P$; the Gibbs free energy equals the Helmholtz free energy, $NE_P/2$. The internal energy, enthalpy and free energy thus scale as N^1, the entropy and heat capacity as N^2, and the temperature as N^{-1}, forming a complete set of thermodynamic state functions.

(8) **Evaporation time t_{ev}.** Including greybody factors and all particle species that can be emitted, the evaporation lifetime of a black hole is $t_{\mathrm{ev}} \approx 6.25\times 10^{-18}\, M^3/f$ s (M in kg), where f is the effective weight of the emitted species, with $f = 15.35$ for small black holes emitting all Standard Model particles [40]. Substituting $M = Nm_P$ with $m_P = 2.176\times 10^{-8}$ kg gives $t_{\mathrm{ev}} \approx 6.44\times 10^{-41} N^3/f$ s; in units of the Planck time $t_P = 5.39\times 10^{-44}$ s, $t_{\mathrm{ev}} \approx (1194/f)\, N^3 t_P$, i.e. $t_{\mathrm{ev}} \approx 77.8\, N^3 t_P$ for $f = 15.35$. For small N, the formulas for the Hawking temperature and the evaporation time are semiclassical extrapolations to the Planck scale.

These results are summarized in Table 1. The maximum force sits at the N^0 rung: whatever the size of the black hole, the ratio of energy to horizon diameter is the same

$c^4/(4G)$, the universal form, along the black-hole sequence, of the maximum force derived from the capsule in Section 3.5. Table 2 gives numerical values for several representative N.

A Schwarzschild black hole has only one independent parameter, its mass M (or energy $E = Mc^2$). With M as the independent variable one obtains only macroscopic functional relations between quantities; with E_c as the counting unit, this parameter becomes the integer $N = M/m_P$, each unit being a capsule with internal structure. The core of Table 1 is that all other physical quantities are dependent variables of N, each equal to the corresponding quantity of a single E_c capsule multiplied by an integer power of N; and because N is an integer, the black-hole entropy can be decomposed capsule by capsule into self terms and pairwise relational terms (Section 4.3), with the sequence starting at $N = 1$ and the smallest black hole at $N = 2$. This is microscopic information that cannot be obtained with M as the independent variable.

Table 1 The N-scaling ladder of the physical quantities of a Schwarzschild black hole

Power of N	Quantity	Expression
N^{-2}	Mean energy density (ρ_H)	$\rho_H = \rho_c/N^2$
N^{-1}	Hawking temperature (T_H)	$k_B T_H = E_P/(8\pi N)$
N^0	Maximum force ($F_{\max}$)	$E/D_H = c^4/(4G) = F_{\max}$
N^1	Internal energy (U), enthalpy, free energy (F), horizon diameter (D_H)	U (= enthalpy) $= NE_P$, $F = NE_P/2$, $D_H = ND_o$
N^2	Horizon area (A_H), entropy (S_H), heat capacity (C)	$A_H = N^2 A_o$, $S_H/k_B = 4\pi N^2$, $C = -8\pi k_B N^2$
N^3	Volume (V_H), evaporation time (t_{ev})	$V_H = N^3 V_c$, $t_{\text{ev}} \approx 77.8 N^3 t_P$ ($f = 15.35$)

Table 2 Numerical values for representative N

Object	N	D_H	ρ_H/ρ_c	T_H	S_H/k_B
E_c (limit point)	1	6.46×10^{-35} m	1	—	$4\pi \approx 12.57$
Smallest black hole	2	1.29×10^{-34} m	0.25	2.82×10^{30} K	$16\pi \approx 50.27$
$1\,M_\odot$	9.14×10^{37}	5.91 km	1.20×10^{-76}	6.17×10^{-8} K	1.05×10^{77}
$10^6\,M_\odot$	9.14×10^{43}	5.91×10^{9} m	1.20×10^{-88}	6.17×10^{-14} K	1.05×10^{89}
$10^9\,M_\odot$	9.14×10^{46}	5.91×10^{12} m	1.20×10^{-94}	6.17×10^{-17} K	1.05×10^{95}
Hubble sphere (Sect. 4.4)	4.25×10^{60}	2.74×10^{26} m	5.55×10^{-122}	—	2.27×10^{122}

T_H is computed from Eq. (32); E_c and the Hubble sphere are not black holes, so no temperature is listed. $M_\odot = 1.989 \times 10^{30}$ kg, $m_P = 2.176 \times 10^{-8}$ kg; for the Hubble sphere $H_0 = 67.4$ km s^{-1} Mpc^{-1} $= 2.1843 \times 10^{-18}$ s^{-1} and $\ell_p = 1.616 \times 10^{-35}$ m. In the Hubble-sphere row, N denotes $N_{\mathcal{H}}$, and D_H, ρ_H/ρ_c and S_H/k_B are the corresponding quantities of the Hubble sphere.

4.3 Relational decomposition of black-hole entropy and the $2d_m$ axiom

Equation (33) gives $S_H/k_B = 4\pi N^2$. Let $P(N,k) = N!/(N-k)!$ denote the number of k-permutations of N distinguishable capsules, i.e. the number of ordered k-tuples of distinct capsules; in particular $P(N,1) = N$ and $P(N,2) = N(N-1)$. Splitting N^2 gives an exact identity:

$$\frac{S_H}{k_B} = 4\pi N^2 = 4\pi \cdot N + 4\pi N(N-1) = 4\pi\,[N + N(N-1)] = 4\pi\,[P(N,1) + P(N,2)] \tag{37}$$

The two terms on the right of Eq. (37) each have a clear meaning. The first is the self term of the N capsules, the single-capsule permutations $P(N,1) = N$, each contributing 4π: the area of a single capsule's outer interface, $A_o = 16\pi\ell_p^2$, corresponds by Eq. (5) to an entropy of exactly $A_o/(4\ell_p^2) = 4\pi$. This 4π also echoes the result of Section 3.5: the capsule energy E_0 is exactly 4π times the minimum string energy $\tau \cdot 2\ell_p$. The second term is the number of ordered pairwise relations among the N capsules, $P(N,2) = N(N-1)$, each relation contributing 4π. The relational terms carry the same weight as the self terms: each added capsule increases the horizon area by $(2N+1)A_o$, which is exactly one A_o for the new self term and one A_o for each of the $2N$ new relations; the self term is itself the relation between the pair of minimal units inside a capsule, and belongs to the same kind of elementary relation as the relations between capsules. The identity has no remainder: the black-hole entropy consists entirely of capsule self terms and ordered pairwise relations, and contains no independent many-body entropy contribution irreducible to pairwise relations.

The self terms do not contradict the statement of Section 2.1 that "an isolated object cannot carry information": what is indivisible is the minimal unit d_m, whereas the equivalent diameter of an E_c capsule is $2d_m$, so the capsule already contains a relation between a pair of minimal units. The self term 4π is this relation already established inside the capsule, the information that E_c carries as the minimal relational structure. According to the first-in-last-out rule of the framework, the capsules inside a black hole are registered and distinguished by the order in which they entered the horizon, and the relations established between them are also identifiable. Moreover, the relations listed in Section 2.1, such as left and right, front and back, above and below, are themselves directed: i relative to j and j relative to i are two distinguishable relations. The total number of identifiable relations must therefore be counted with permutations rather than combinations.

Conversely, if there were three-body relations among the capsules that cannot be reduced to pairwise ones, $P(N,3) = N(N-1)(N-2) \propto N^3$, the entropy would contain a term proportional to the volume, in contradiction with the area scaling of Eq. (5) and the holographic principle [27–29]. Hence "relations are established only between pairs of minimal units" and "entropy is proportional to the horizon area" are

two statements of the same fact:

$$\begin{array}{c}\text{relations only between pairs of units}\\ \Updownarrow\\ \text{number of relations} \propto N^2\\ \Updownarrow\\ \text{entropy} \propto \text{horizon area}\end{array}$$

This is exactly the content of the $2d_m$ axiom. Equation (37) also shows that, for black holes, the entropy formula can only hold as an equality. Each added capsule increases the horizon area by exactly $(2N+1)A_o$, corresponding to one self term and $2N$ new relations, each occupying one A_o; all self terms of the capsules in the black hole and all their pairwise relations are established and counted, with no vacancies, and the entropy equals the sum of all these terms, $4\pi k_B N^2$, i.e. the equality in Eq. (5). A strict inequality would require some capsules, or some pairwise relations between them, not to be established, contradicting the fact that the capsules in a black hole are distinguished by the order in which they entered the horizon and all their pairwise relations are identifiable. The inequality form of entropy bounds applies only to other systems, in which not all relations are established: for example, N background E_c capsules with no relations among them carry only self terms, with entropy $4\pi k_B N$, far below the entropy $4\pi k_B N^2$ of a black hole of the same energy.

Sections 2.1–2.2 started from "information is a measure of distinguishable relations between objects" and defined $2d_m$ as the minimum span required for two indivisible minimal units to establish a relation for the first time: the elementary unit of a relation is a pair of units. Equation (37) shows that in the system of highest information density in the universe, the black hole, the entropy decomposes in exactly this way: apart from the self term of each unit, everything comes from pairwise relations, with no many-body remainder. The fraction of the total entropy carried by the relational terms is $\eta = 4\pi N(N-1)/(4\pi N^2) = (N-1)/N$. For $N = 1$, $\eta = 0$ and only the self term exists; relational terms appear for the first time at $N = 2$, where $\eta = 0.5$, i.e. they make up half of the total entropy, which corroborates the statement in Section 4.1 that black holes start at $N = 2$; for a solar-mass black hole ($N \sim 10^{38}$), $\eta \approx 1$, and the entropy comes almost entirely from relations. Section 2.4 used the Bekenstein–Hawking entropy as a cross-scale corroboration of the $2d_m$ axiom, establishing the correspondence between the minimum area cell $4\ell_p^2$ and 1 bit; Eq. (37) shows further that the overall structure of black-hole entropy likewise obeys "information is relation".

Equation (37) rests on the Schwarzschild solution and Eq. (29); the horizon area of a rotating black hole depends on its spin, and the corresponding decomposition is left for future work. Equation (37) is an algebraic identity and does not by itself prove the $2d_m$ axiom; what it shows is that, when counted in units of E_c, the structure of black-hole entropy is fully compatible with the pairwise relations described by the axiom.

4.4 Connection with the cosmological residual potential

The $1/N^2$ scaling of Eq. (31) also appears on the cosmological scale. For a spatially flat universe, the Friedmann equation gives the critical (mass) density $\rho^{\mathrm{m}}_{\mathrm{crit}} = 3H^2/(8\pi G)$. Consider a sphere whose radius is the Hubble radius $R_{\mathcal{H}} = c/H$. The mass it contains is

$$M_{\mathcal{H}} = \frac{4\pi}{3} R_{\mathcal{H}}^3 \, \rho^{\mathrm{m}}_{\mathrm{crit}} = \frac{4\pi}{3} \left(\frac{c}{H}\right)^3 \cdot \frac{3H^2}{8\pi G} = \frac{c}{H} \cdot \frac{c^2}{2G} = \frac{R_{\mathcal{H}} c^2}{2G}$$

and therefore

$$R_{\mathcal{H}} = \frac{2GM_{\mathcal{H}}}{c^2} = R_s(M_{\mathcal{H}}) \tag{38}$$

where $2GM_{\mathcal{H}}/c^2 = R_s(M_{\mathcal{H}})$ is the Schwarzschild radius of a black hole of mass $M_{\mathcal{H}}$: the Hubble sphere lies exactly at its own Schwarzschild radius, a known property of a flat universe. $N_{\mathcal{H}}$ is the equivalent capsule count of the total mass–energy of the Hubble sphere ($M_{\mathcal{H}} c^2 = N_{\mathcal{H}} E_P$). The Hubble sphere, however, is not a black hole: its interior is filled with background capsules, whose actual number is $\phi V_{\mathcal{H}}/V_c = \phi N_{\mathcal{H}}^3 \sim 10^{181}$ (ϕ see Section 6); their energy far exceeds $M_{\mathcal{H}} c^2$ but, being in zero-sum balance, does not count toward the gravitating mass–energy (Section 5.3). The ratio of the two counts, $\phi N_{\mathcal{H}}^2 \sim 10^{121}$, is in order of magnitude the gap between the quantum-field-theory estimate of the vacuum energy and observation: the former counts all background capsules, the latter only the gravitating mass–energy. More precisely, the ratio of the background energy to the observed dark energy is $E_{\mathrm{bg}}/E_\Lambda = \phi N_{\mathcal{H}}^2/\Omega_\Lambda = 1/\delta_{\mathrm{bg}}$ (δ_{bg} see Section 6). Viewed as a black hole, the Hubble sphere appears only as the gravitating mass–energy $N_{\mathcal{H}} E_P$; background capsules, in zero-sum balance, do not gravitate. The microscopic capsule count and the macroscopic gravitating mass–energy are thereby connected. We may therefore treat the Hubble horizon (here meaning the Hubble sphere of radius $R_{\mathcal{H}}$, not the cosmological event horizon) as the case $N = N_{\mathcal{H}}$ and apply Eqs. (28) and (31):

$$N_{\mathcal{H}} = \frac{R_{\mathcal{H}}}{2\ell_p} = \frac{c}{2H\ell_p}, \qquad \rho^{\mathrm{m}}_{\mathrm{crit}} \, c^2 = \frac{\rho_c}{N_{\mathcal{H}}^2} \tag{39}$$

Taking $H = H_0 = 67.4 \ \mathrm{km\,s^{-1}\,Mpc^{-1}} = 2.1843 \times 10^{-18} \ \mathrm{s^{-1}}$ and $\ell_p = 1.616 \times 10^{-35}$ m gives $N_{\mathcal{H}} \approx 4.246 \times 10^{60}$ and $N_{\mathcal{H}}^2 \approx 1.803 \times 10^{121}$. The dark-energy density ρ_{obs} is $\Omega_\Lambda \approx 0.685$ times the critical density (the critical mass density times c^2), so

$$\rho_{\mathrm{obs}} = \Omega_\Lambda \, \rho^{\mathrm{m}}_{\mathrm{crit}} \, c^2 = \frac{\Omega_\Lambda}{N_{\mathcal{H}}^2} \, \rho_c = \delta_c \, \rho_c = \frac{3}{32\pi} \, \delta_c \, \rho_P = \delta_P \, \rho_P = 5.25 \times 10^{-10} \ \mathrm{J\,m^{-3}} \tag{40}$$

Here the observed dark-energy density ρ_{obs} is the cosmological residual potential (Section 5.3); its dimensionless residual coefficient relative to the capsule energy density is $\delta_c = \Omega_\Lambda/N_{\mathcal{H}}^2 = 0.685 \times (5.55 \times 10^{-122}) \approx 3.80 \times 10^{-122}$, and the residual coefficient relative to the Planck energy density is $\delta_P = [3/(32\pi)] \, \delta_c \approx 1.13 \times 10^{-123}$.

The coefficient δ_c is constant if, and only if, ρ_{obs} is constant, i.e. in ΛCDM. There $\Omega_\Lambda = \rho_{\mathrm{obs}}/(\rho^{\mathrm{m}}_{\mathrm{crit}} c^2) \propto H^{-2}$, while $N_{\mathcal{H}}^2 \propto H^{-2}$ by Eq. (39), so the ratio $\Omega_\Lambda/N_{\mathcal{H}}^2$ does not change even though Ω_Λ and $N_{\mathcal{H}}$ each evolve. We adopt this setting, consistent with current observations. It has a simple meaning. In a flat ΛCDM universe (neglecting

radiation) $H^2 = H_0^2(\Omega_m a^{-3} + \Omega_\Lambda)$, so as $a \to \infty$ the expansion rate tends to $H_\infty = H_0\sqrt{\Omega_\Lambda}$ and the universe approaches a de Sitter state, in which the Hubble sphere coincides with the cosmological event horizon. The gravitating mass–energy equivalent capsule count of this asymptotic Hubble sphere is $N_\infty = c/(2H_\infty \ell_p) = N_\mathcal{H}/\sqrt{\Omega_\Lambda} \approx 5.13 \times 10^{60}$ (with the present values of $N_\mathcal{H}$ and Ω_Λ), so that the residual coefficient of Eq. (40) takes the form

$$\delta_c = \frac{1}{N_\infty^2} \tag{41}$$

In this limit $\Omega_\Lambda \to 1$ and $N_\mathcal{H} \to N_\infty$, and $\delta_c = 1/N_\mathcal{H}^2$ takes exactly the $1/N^2$ form of the mean density of a black hole (Eq. (31)); its value is taken from observation.

5 Full topological normalization and resolution of the vacuum energy density paradox

5.1 Background and the three classical derivation paths

As stated in the Introduction, the vacuum zero-point energy density estimated with the Planck wave-vector cutoff k_p exceeds the cosmologically observed value by 120–123 orders of magnitude.

In the standard derivation, the Planck scale forcibly exposes three fundamental physical paradoxes:

1. **Substance vs. vacuum.** The extreme substance of maximally collapsed matter (a black hole) and the ground state without real particles (the vacuum) yield energy densities that overlap completely in magnitude.
2. **Upper bound vs. lower bound.** The maximum mass–energy that the spacetime manifold can carry overlaps completely with the minimum energy that quantum fluctuations must possess.
3. **Gravity vs. repulsion.** The strong gravity that bends spacetime severely and drives collapse inward, and the vacuum zero-point repulsion that drives the accelerated expansion of the universe outward, are assigned the same enormous value (10^{113} J m^{-3}).

Before any topological correction, the energy densities computed along the three classical paths are as follows.

Path A (the continuous-field ground-state approach of quantum field theory) [1, 2]:

$$\rho_{\text{vac}}^A = \int_0^{k_p} \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(\frac{1}{2}\hbar c k\right) = \frac{1}{16\pi^2} \cdot \frac{\hbar c}{\ell_p^4} \approx 0.00633\, \rho_P \tag{42}$$

Path B (the Planck-scale mass–energy limit of gravitational collapse) [41, 42]:

$$\rho_{\text{vac}}^B = \frac{E_P}{V_P} = \frac{m_P c^2}{\ell_p^3} = \frac{\hbar c}{\ell_p^4} = 1.0\, \rho_P \tag{43}$$

Path C (normalization of the single-mode zero-point energy of microscopic quantum mechanics):

$$\rho_{\rm vac}^{C} = \frac{(1/2)\hbar\omega_p}{\ell_p^3} = \frac{1}{2}\cdot\frac{\hbar c}{\ell_p^4} = 0.5\,\rho_P \tag{44}$$

The result of Path A is about two orders of magnitude lower than those of Paths B and C; the difference comes from the momentum-sphere integration factor $1/(16\pi^2)$. Path A treats momentum space as continuous and only imposes a cutoff at $k_p = 1/\ell_p$; in the present framework the minimal structure is the discrete capsule (Section 3), and energy is counted per capsule rather than integrated over continuous modes, so the factor $1/(16\pi^2)$ does not apply at the capsule scale.

5.2 Full topological normalization based on V_c and the exact algebraic equality $\rho_c^B = \rho_c^C$

Section 3.1 gave the fully topologically normalized volume of the capsule (Eq. (9)), $V_c = (32\pi/3)\ell_p^3$. Substituting it into the three paths, the normalization proceeds as follows: Paths A and B first take the energy within one Planck cube ℓ_p^3 ($\rho_{\rm vac}^A\ell_p^3 = m_Pc^2/(16\pi^2)$ and m_Pc^2, respectively) and then distribute it over one capsule volume V_c, i.e. each is multiplied by $\ell_p^3/V_c = 3/(32\pi)$; Path C counts the total zero-point energy of the capsule, $2E_m = \hbar\omega_p$ (Section 3.4, two minimal units), and divides it by V_c:

$$\rho_c^A = \frac{3}{32\pi}\cdot\frac{1}{16\pi^2}\cdot\frac{\hbar c}{\ell_p^4} = \frac{3}{512\pi^3}\cdot\frac{\hbar c}{\ell_p^4} \approx 0.000189\,\rho_P \tag{45}$$

$$\rho_c^B = \frac{m_Pc^2}{V_c} = \frac{3}{32\pi}\cdot\frac{\hbar c}{\ell_p^4} \approx 0.02984\,\rho_P \tag{46}$$

$$\rho_c^C = \frac{2E_m}{V_c} = \frac{\hbar\omega_p}{V_c} = \frac{3}{32\pi}\cdot\frac{\hbar c}{\ell_p^4} \approx 0.02984\,\rho_P \tag{47}$$

After normalization, Path C (the quantum zero-point-field view) and Path B (the gravitational mass–energy view) thus reach an exact 1:1 algebraic equality:

$$\rho_c^B = \rho_c^C = \frac{3}{32\pi}\cdot\frac{\hbar c}{\ell_p^4} \approx 0.02984\,\rho_P \tag{48}$$

Equation (48) embodies the meeting of the gravitational and quantum descriptions at the capsule scale, which is precisely the physical content of the Planck scale, and it has definite positions in the capsule structure: the Schwarzschild radius of the Planck mass, $2Gm_P/c^2 = 2\ell_p$, is exactly the outer-interface radius R_o (gravity side, Path B), and its reduced Compton wavelength, $\hbar/(m_Pc) = \ell_p$, is exactly the inner-interface radius R_i (quantum side, Path C). The factor 2 that makes the two exactly equal comes from the $2d_m$ axiom (Section 3.4). Because space must in fact begin with the spherical region of outer radius $R_o = 2\ell_p$, $V_c = (32\pi/3)\ell_p^3$, the Planck density itself cannot be realized in a sphere of radius ℓ_p, $(4/3)\pi\ell_p^3$; yet much of the literature still treats the Planck density as the upper bound on energy density. Our capsule-unit density ($\rho_c = \rho_c^B = \rho_c^C$)

is the physically meaningful upper bound on density. The existence of this actual upper bound also means that spatial contraction cannot approach zero indefinitely and density cannot diverge, which provides a natural physical regularization mechanism for the black-hole singularity problem. Nonsingular black-hole models constructed under the limiting-curvature hypothesis [43] are consistent with this direction.

5.3 Resolution of the paradoxes and the dark-energy phase-transition residual potential

The exact algebraic equality $\rho_c^B = \rho_c^C$ (Eq. (48)) resolves, directly at the phase-transition interface, the three classical oppositions of Section 5.1: substance vs. vacuum, upper bound vs. lower bound, and gravity vs. repulsion. Path C corresponds to the inner side: at the inner interface $R_i = \hbar/(m_P c) = \ell_p$, the zero-point energy inside the capsule pushes outward. Path B corresponds to the outer side: at the outer interface $R_o = 2Gm_P/c^2 = 2\ell_p$, the spacetime interfacial tension pulls inward. Equation (48) makes the two equal; this is precisely the force balance of Section 3.5: the outward push of the capsule equals the total pull $F_{\max}$ of its inner and outer interfaces (Eq. (26)). Paths B and C are two sides of one capsule balance, not two superposed fluids. The equality reveals the Planck scale as the critical phase-transition interface between the implicate order (the non-local topological vacuum) [44, 45] and the explicate order (local metric spacetime) [44, 45]: the inward gravitational binding of the explicate order (Path B, the Planck substance limit) and the outward topological zero-point repulsion of the implicate order (Path C, the single-sided zero-point mode) reach an exact 1:1 dynamical balance at the interface.

Energy density is an intensive quantity. The vacuum zero-point energy computed in quantum field theory with the Planck cutoff is itself a density on the microscopic scale. Coarse-graining over volumes much larger than V_c, and since all capsules are identical, the background mean density is $\phi\rho_c$ (ϕ being the packing fraction of the capsules, between $\pi/6$ and 1; see Section 6), which differs from the capsule-scale comparison only by a factor of order unity. Inside each capsule, the outward push of the zero-point energy and the inward pull of the interfacial tension balance according to Eq. (48), and the capsule is a closed, self-balanced unit: the force balance ensures its stability, and its positive zero-point energy and the negative gravitational energy associated with the spacetime interface cancel algebraically (the zero-sum balance of Section 6), so that as a whole it is not a macroscopic source of gravity. This is in line with the idea of a universe of zero total energy, in which positive matter energy and negative gravitational energy cancel [46]. Only the residual departing from this balance acts gravitationally on macroscopic scales.

The observed dark-energy density ($\rho_{\text{obs}} \approx 5.25 \times 10^{-10}\ \text{J m}^{-3}$) [3–5] is not the absolute zero-point energy itself, but the small, non-equilibrium topological phase-transition residual potential left behind as explicate-order spacetime deviates from the ideal symmetry point during its holographic unfolding; its magnitude relative to the Planck energy density is $\delta_P \approx 1.13 \times 10^{-123}$ (Eq. (40)). The enormous Planck energy density, acting as the background tension of the phase transition, is cancelled completely at the interface, while the small residual potential left over ($\delta_P \cdot \rho_P$) acts as a net thermodynamic potential difference that keeps driving the accelerated expansion

shared by the implicate-order background field and the explicate-order foreground spacetime.

Causal-set theory obtains $\rho \sim \pm\alpha(Ht_P)^2\rho_P$ from Poisson fluctuations in the number of spacetime elements [8, 9] (t_P is the Planck time and α an amplitude fitted to observations); that is, the dark-energy density is proportional to $(Ht_P)^2$. From Eq. (39), $N_{\mathcal{H}} = c/(2H\ell_p) = 1/[2H(\ell_p/c)] = 1/(2Ht_P)$, so Eq. (40) of this paper can be rewritten as $\rho_{\rm obs} = [3/(8\pi)]\,\Omega_\Lambda(Ht_P)^2\rho_P$. The difference between the two is this. In causal sets the residual comes from random fluctuations, can be positive or negative, evolves as H^2, and is of the order of the critical density at all epochs. In this paper the residual is the net difference between two opposing quantities, Paths B and C, near the exact symmetry point, as in a tug-of-war in which one side holds a slight advantage. Unlike a fluctuation, this net difference has a definite sign; the observed $\Omega_\Lambda > 0$ shows that the outward side (Path C) holds the slight advantage. With δ_c constant (Eq. (41)), $\rho_{\rm obs} = \delta_c\rho_c$ is negligible compared with matter and radiation at early times, and the residual potential always drives the universe to expand in one direction. These differences can in principle be distinguished by observations of the early universe and of the dark-energy equation of state.

6 Outlook: continuous creation of background capsules

Sections 4 and 5 take the capsule energy density ρ_c (a constant) as the reference, with the observed dark-energy density as the residual $\delta_c\rho_c$. Keeping the background mean density undiluted as the universe expands requires the continual creation of background capsules. The following is stated as a working hypothesis.

We propose a hypothesis: cosmic expansion may proceed globally through the dark-energy capsule units E_c adopting a mode similar to biological cell division, creation being based on the constant capsule density ρ_c ($= \rho_c^B = \rho_c^C$). The background mean density is $\phi\rho_c$, where ϕ is the packing fraction of the capsules, which depends on how the E_c pack and is not yet known. We conjecture that it lies between simple cubic packing (each capsule occupying a cubic cell of side $4\ell_p$, $\phi = V_c/(4\ell_p)^3 = \pi/6 \approx 0.524$) and equivalent-cube packing (each capsule equivalent to a cube of volume V_c, filling space, $\phi = 1$), a range that includes the close packing of equal spheres ($\phi = \pi/(3\sqrt{2}) \approx 0.740$); the background mean density thus lies between $(\pi/6)\rho_c$ and ρ_c. Creation keeps the capsule number density $n = \phi/V_c$ constant. Correspondingly, the residual coefficient relative to the background mean density is $\delta_{\rm bg} = \rho_{\rm obs}/(\phi\rho_c) = \delta_c/\phi$. Each E_c capsule is created continuously, locally and independently, with a constant energy per capsule ($2E_m = \hbar\omega_p$), without reference to any global centre or external coordinate system. In this replication, a newly created E_c capsule does not draw energy from pre-existing space; instead, the negative gravitational potential energy brought by the topological-geometric creation of space and the positive zero-point energy introduced by the new capsule ($2E_m = \hbar\omega_p$) reach a strict algebraic zero-sum balance. This mechanism of "unchanged capsule size, undiminished energy per capsule, increasing global number" ensures that the background mean density $\phi\rho_c$ is not diluted as space expands, and is consistent with the dark-energy equation of state $w \equiv p/\rho \approx -1$.

It should be noted that the idea of "continuous creation to keep the density constant" is not new in the history of cosmology. It resembles in mechanism the steady-state cosmology proposed by Bondi and Gold [47] and by Hoyle [48], which has been ruled out by observational evidence such as the cosmic microwave background discovered by Penzias and Wilson [49]. The key difference between the present hypothesis and the steady-state theory is that the creation here is confined to the vacuum-energy-density component alone, rather than requiring the continuous creation of all matter.

6.1 A falsifiable creation-rate equation

To make the simplest form of this hypothesis testable, we give the following creation-rate equation. Let ρ_c be the capsule energy density (a constant; the observed dark-energy density is the residual potential $\delta_c\rho_c$, see Eq. (40); and $\rho_c^B = \rho_c^C$), ϕ the packing fraction, so that the background mean density is $\phi\rho_c$, and let $V(t)$ be the physical volume of a fixed comoving region, whose expansion rate satisfies $\mathrm{d}V/\mathrm{d}t = 3HV$ (since $V \propto a^3$). For the background mean density $\phi\rho_c = E_{\mathrm{total}}/V$ to stay constant, the added energy must match the growth of volume exactly:

$$\frac{\mathrm{d}(\phi\rho_c V)}{\mathrm{d}t} = \phi\rho_c \frac{\mathrm{d}V}{\mathrm{d}t} = 3H\phi\rho_c V \tag{49}$$

If all of this added energy comes from newly created E_c capsules (each contributing $2E_m = \hbar\omega_p$), the capsule creation rate satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{3H\phi\rho_c V}{2E_m} = \frac{3H\phi\rho_c V}{\hbar\omega_p} = \frac{3H\phi V}{V_c} \tag{50}$$

where $\mathrm{d}N/\mathrm{d}t$ is the number of E_c capsules newly created per unit time, H is the Hubble parameter and V is the physical volume of a fixed comoving region. This specific quantitative relation can in principle be compared with future, more precise observations of the dark-energy equation of state $w(z)$, offering a way to test or falsify the hypothesis. Background-capsule creation is based on a constant ρ_c; whether δ_c is strictly constant is decided by observation. The simplest form (δ_c constant) predicts $w = -1$; if $w(z)$ evolves, as hinted by DESI [4, 5], δ_c varies accordingly. If the residual is conserved separately as an effective dark-energy component, then from $\rho_\Lambda = \delta_c\rho_c$ one has $\dot{\rho}_\Lambda = \dot{\delta}_c\rho_c$ (ρ_c constant), and substitution into the continuity equation $\dot{\rho}_\Lambda + 3H(1+w)\rho_\Lambda = 0$ gives

$$w = -1 - \frac{\dot{\delta}_c}{3H\delta_c} \tag{51}$$

Constant δ_c gives $w = -1$; decreasing δ_c gives $w > -1$. Because what is created is background capsules, ρ_c and the packing fraction ϕ both remain constant; δ_c is constant in ΛCDM (Eq. (41)), consistent with current observations; this background creation is a process distinct from the packing of capsule energy into black holes by accretion and its unpacking in bursts.

7 Conclusions

1. The common-sense principle of the relational nature of information independently yields the $2d_m$ axiom, which is exactly self-consistent with black-hole entropy and the holographic principle under extreme physical conditions; this provides the axiom with an independent, cross-scale theoretical corroboration.
2. Applying the $2d_m$ axiom independently to the dark-energy capsule unit and to massless particles, their equivalent diameters coincide naturally at the level of the axiom ($2\ell_p$; Eqs. (7) and (10)). Further energy accounting shows that the capsule's zero-point energy and the energy ceiling of massless particles likewise coincide exactly at the Planck energy E_P, a non-trivial result in which three minimum quantities jointly lock in a single upper bound (Eq. (14)).
3. Boundary constraint of the capsule: interpreting the reciprocal of the field-equation coupling constant, $\tau = c^4/(8\pi G)$, as a line energy density on spacetime interfaces gives the spacetime interfacial tension $\sigma = \tau/(2R)$ and a size-independent interfacial pull $\pi\tau$ (Eq. (17)); the total pull of the capsule's inner and outer interfaces is $c^4/(4G)$, exactly the maximum-force conjecture value, obtained without assuming the conjecture (Eq. (20)). The balance between the outward push and the total pull gives the equilibrium radius ℓ_p, with the outer interface at the Schwarzschild radius $2\ell_p$, both consistent with the $2d_m$ axiom (Eqs. (24) and (25)); at equilibrium the push equals the maximum force (Eq. (26)), and the equilibrium is stable.
4. With E_c as the counting unit, a Schwarzschild black hole containing N capsule energies satisfies $D_H = N \cdot D_o$ (Eq. (29)), and its quantities form a ladder of integer powers of N (Table 1), the integer N being the only independent variable and all other quantities its dependent variables: mean density $\propto N^{-2}$, temperature $\propto N^{-1}$, maximum force $\propto N^0$, internal energy, enthalpy, free energy and diameter $\propto N$, area, entropy and heat capacity $\propto N^2$, volume and evaporation time $\propto N^3$; E_c is the limit point $N = 1$, and the smallest black hole consists of two capsules.
5. Black-hole entropy satisfies the exact identity $S_H/k_B = 4\pi N^2 = 4\pi[P(N,1) + P(N,2)]$ (Eq. (37)): the sum of the capsule self terms plus the number of ordered pairwise relations times 4π, with no many-body remainder. This decomposition is fixed by the linear superposition of horizon diameters, is equivalent to the holographic area law, and corresponds strictly to the statement of the $2d_m$ axiom that "relations are established between pairs of minimal units"; the relational terms make up $(N-1)/N$ of the total entropy and first appear at $N = 2$. Equation (37) also shows that for black holes the entropy formula can only hold as an equality: every capsule's self term and every pair's relational term are established and fully counted, and the black hole is a completely saturated system in which all capsules and all their pairwise relations are identified; the inequality form of entropy bounds applies only to other systems, in which not all relations are established.
6. Based on the full topological normalization over the capsule volume, Path B (the gravitational mass–energy limit) and Path C (the quantum zero-point field) reach an exact 1:1 algebraic equality (Eq. (48)): Path C (inner side, zero-point push) and Path B (outer side, interfacial-tension pull) are two sides of one capsule

balance, i.e. the force balance of Section 3.5; this resolves the three classical paradoxes of "substance vs. vacuum", "upper bound vs. lower bound" and "gravity vs. repulsion".

7. Within this framework, the positive zero-point energy and the negative gravitational energy of the background capsules cancel in a zero sum, so that as a whole they are not a macroscopic source of gravity. The observed dark-energy density is reinterpreted as the small, non-equilibrium topological phase-transition residual potential left, during the holographic unfolding of explicate-order spacetime, relative to the ideal symmetry point at which Paths B and C cancel exactly ($\rho_c^B = \rho_c^C$). Its coefficient (δ_c) obeys the same $1/N^2$ law as the mean density of black holes, $\delta_c = \Omega_\Lambda / N_{\mathcal{H}}^2$ (Eq. (40)), where $N_{\mathcal{H}} \approx 4.25 \times 10^{60}$ is the gravitating mass–energy equivalent capsule count of the Hubble sphere (Eq. (39)); the ratio to it of the background-capsule number in the Hubble sphere, $\phi N_{\mathcal{H}}^3 \sim 10^{181}$, namely $\phi N_{\mathcal{H}}^2 \sim 10^{121}$, is the origin of the gap of about 120 orders of magnitude between the quantum-field-theory estimate and observation; in ΛCDM, $\delta_c = 1/N_\infty^2$, where N_∞ is the gravitating mass–energy equivalent capsule count of the asymptotic de Sitter Hubble sphere (Eq. (41)). The existence of the capsule-unit density as an actual upper bound on physical density also offers a regularization picture for the black-hole singularity problem that merits further study.
8. As an outlook, the working hypothesis of continuous local creation of background capsules E_c, based on a constant capsule density ρ_c and keeping the capsule number density constant, keeps the background mean density $\phi\rho_c$ undiluted and yields a creation-rate equation tied to the Hubble parameter, a kinematic consequence of constant number density. The residual coefficient and the equation of state satisfy $w = -1 - \dot{\delta}_c/(3H\delta_c)$ (Eq. (51)); the simplest form, with δ_c constant (ΛCDM, consistent with current observations), predicts $w = -1$, testable by future measurements of $w(z)$.

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